LOCALISATIONS OF NON-PRIME ORDERS

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DECLARATION

The material in this thesis is original except where indicated otherwise. Chapters two to six are the work of the author and a synopsis has been accepted by Communications in Algebra for publication. The special case of maximal orders in Artinian quotient rings of Chapter seven was the joint work of the author and Dr. C. Hajarnavis and a paper on this will appear in the Journal of Algebra. The proofs given there have now mainly been shortened.

ABSTRACT

This thesis is the study of non-prime orders and their localisations. It generalises the work of Chamarie, Marubayashi and Fujita on prime Goldie maximal orders and e-ν H-orders, as well as moving in new directions. In Chapter two we introduce the notion of an additive regular ring demonstrating their importance in non-prime orders and proving C(A) being an Ore set $\Rightarrow$ $C(A) \cap C(0) = S(A)$ is an Ore set for A an R-ideal of R. This leads us to consider both the rings R_P and $R_{S(P)}$ (which both coincide on the prime case). In Chapter three we introduce three chain conditions on an e-ν H-order namely τ-Noetherian, r-τ-Noetherian, τ_o-Noetherian and use these to give circumstances for when C(A) or S(A) is an Ore set. In Chapter four we look at localisations of e-νH-orders showing they are νH-orders and discuss the problem of when they have enough ν-invertible ideals. In Chapter five we look at the structure of $R_{S(A)}$ for ν-invertible ideals A when R has a semi-local quotient ring, and give an intersection theorem analogous to the prime case. In Chapter six we look at the stu cture of R_P for P a maximal ν-invertible ideal and discuss it s rank under various chain conditions. In Chapter seven we prove a splitting theorem for τ_o-Noetherian e-νH-orders in Artinian quotient rings and give applications. In Chapter eight we look at the structure of e-νH-orders with a finite number of ν-idempotent ideals. Finally in Chapter nine examples are given to show the theory is not redundant and further problems are discussed.

TO THE MEMORIES OF MY MOTHER AND AUNT

'The hills step off into whiteness.
People or stars
Regard me sadly, I disappoint them.

The train leaves a line of breath.
O slow
Horse the colour of rust,

Hooves, dolorous bells -
All morning the
Morning has been blackening,

A flower left out.
My bones hold a stillness, the far
Fields melt my heart.

They threaten
To let me through to a heaven
Starless and fatherless, a dark water.'

Sylvia Plath - Sheep in fog.

INTRODUCTION

This thesis is the study of non-prime orders. We first put things in an historical context. The study of prime Goldie maximal orders was revitalised by the paper [15] where Chamarie showed that reflexive prime ideals of a Noetherian maximal order are localisable and

$R = \bigcap_{P \in \mathcal{P}} R_p \cap S(R)$, where $\mathcal{P}$ is the collection of maximal reflexive ideals of R. S(R) = Asano overring.

Also R_p is a local hereditary ring. (*)

In the important paper [14] Chamarie weakened the Noetherian chain condition and introduced a new class of maximal orders which he called Krull orders which generalise commutative Krull domains and in which the reflexive ideals are localisable and (*) still holds. At the same time as this Marubayashi was looking at those rings satisfying (*) for which S(R) is a simple Noetherian ring. These generalise prime Noetherian Asano orders, see the paper [31]. Interest also started in generalising the R_p in (*) from local hereditary rings to the weaker semi-local hereditary rings, see [30] and [21]. Work on these rings was done by Fujita and Marubayashi giving rise to νHC-orders with enough ν-invertibles and generalised Krull orders. The νHC-orders have the distinct advantage in that they cover both H.N.P. rings and maximal orders. The author will refer to νHC-orders with enough ν-invertibles as τ-Noetherian e-νH-orders. As many maximal orders are not prime rings the next natural development would be the study of non-prime maximal orders and non-prime e-νH-orders.

The first steps were taken by Robson and Hajarnavis in [26] which considered fully bounded Noetherian maximal orders in Artinian quotient rings, where they showed that such rings are the direct sum of prime Noetherian maximal orders and Artinian rings. This was fully generalised by the author and Hajarnavis in [27] to show that Noetherian maximal orders in Artinian quotient rings split as the direct sum of prime maximal orders and a ring S which has no reflexive ideals. A generalisation of this work appears in Chapter seven. A study of non-prime orders in the spirit of [30] and [21] appears in Chapters two to six. In Chapter two we introduce the notion of an additive regular order which enables us to overcome the technical difficulties produced when we drop the prime condition on R. They more importantly have the property that if A is an ideal of R containing a regular element and C(A) is an Ore set then C(A) ∩ C(0) is an Ore set. It should be emphasized that without this new result the study of non-prime orders can go nowhere. It is new even for Noetherian orders which Stafford has recently shown have semi-local quotient rings and hence are additive regular. This leads us to consider the existence of the rings R_P and $R_{S(P)}$ and we find that there are two natural chain conditions to put on R namely τ-Noetherian and r-τ-Noetherian (see Chapter three). The latter condition gives C(A) ∩ C(0) as an Ore set for ν-invertible R-ideals A which need not be localisable. An example of this happening is Small's example $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix} = R$, an r-$\tau$-Noetherian maximal order which is not τ-Noetherian and has no localisable reflexive ideals, but S(P)=C(P) ∩ C(0) is an Ore set for reflexive R-ideals P.

Given a τ-Noetherian maximal order with a semi-local quotient ring we see as in the prime case that every reflexive prime ideal P is localisable but the rank (being one) and the structure of R_P (being local Noetherian and hereditary) is far from clear and one has to go via $R_{S(P)}$ to prove this (Chapter six). In Chapter five we look at the structure of $R_{S(P)}$ proving it is an r-hereditary, regular, semi-local ring and give the natural generalisation of (*) for r-τ-Noetherian e-νH-orders in the non-prime case. In Chapter eight we look at idealisers and in Chapter nine we give examples illustrating the theory and mention some open problems.

CHAPTER ONE - PRELIMINARIES

All rings will be assumed to have an identity, but are not necessarily commutative. When appropriate the lack of the prefix left or right will mean left-right symmetry, e.g. R Noetherian will mean R is both left and right Noetherian. This work will be 'right orientated' in that we will assume all modules to be right modules unless stated otherwise, we will denote the category of right (left) R-modules by mod-R(R-mod); also the proofs of 'two-sided' propositions will always be proved on the right. We will assume the reader is familiar with basic definitions such as right (left) ideals, prime, semi-prime, Noetherian, Artinian etc. See [S] for further details.

§A. Goldie rings

If S is a subset of a ring R or of a right module M then the *right annihilator ideal* of S in R is $r_R(S) = \{x \in R \mid sx = 0 \text{ for all } s \in S\}$. A submodule K of M is said to be *essential* in M if every non-zero submodule of M intersects non-trivially with K. $M \neq 0$ is said to be *simple* if it has no proper submodules. M is *semi-simple* if it is the direct sum of simple modules. $M \neq 0$ is *uniform* if every non-zero submodule of M is essential in M. A module M has *finite uniform dimension* if it does not contain a direct sum of an infinite number of non-zero submodules. A right module M is said to be *(right) Goldie* if it has finite uniform dimension and R satisfies the (a.c.c) ascending chain condition on right annihilators of subsets of M. The ring R is *right Goldie* if R_R is right Goldie. R is said to be *simple* if it has no non-zero ideals. R is *semi-simple* if R_R and ${}_RR$ are semi-simple. A right module M is *faithful* if ann $M_R = 0$. An element c of R is regular if $r(c) = \ell(c) = 0$. An ideal with no regular elements is called a *zero divisor ideal*. An element c of R is a *unit* or *invertible* if there exists $e \in R$ with $1 = ec = ce$.

Proposition 1.1 (Goldie's Theorem)

A ring R is semi-prime Goldie iff it has a semi-simple Artinian quotient ring.

Proof

See Chapter two, 2.2 of [9]. For the definition of quotient rings see §E. □

Corollary 1.2 (Robson)

If R is a semi-prime Goldie ring and I is an essential right ideal of R and $a \in R$, then the set $(a + I)$ contains a regular element.

Proof

See 1.18 of [2]. The case $a = 0$ is due to Goldie. □

Corollary 1.3

If R is a prime Goldie ring and contains a simple right ideal then R is a simple Artinian ring.

Proof.

See 1.24 of [2]. □

§B. The Jacobson radical and Artinian rings

The Jacobson radical of a ring R which we will always denote by J(R) or J is the intersection of the maximal right ideals.

Proposition 1.4

J(R) is the intersection of the maximal left ideals and is the largest ideal of R consisting of elements r such that 1-r is invertible.

Proof

This is 1.5 and 1.6 of Chapter 8 of [9]. □

We say a ring R is *local (semi-local)* if $\frac{R}{J(R)}$ is simple (semi-simple) Artinian. If this happens then R has a unique maximal ideal (finite number of maximal ideals).

Proposition 1.5 (Wedderburn-Artin Theorem)

A ring is semi-simple Artinian iff it is isomorphic to a direct sum of matrix rings over division rings.

Proof

See 13.4 of [1]. □

If R is a Noetherian ring then the *Artinian radical* of R denoted by A(R) is the sum of the Artinian right ideals of R. By 4.9 of [2] it is also the sum of the Artinian left ideals of R. A(R) is an ideal of R.

<u>Theorem 1.6</u> (Ginn and Moss)

If R is a Noetherian ring with an Artinian quotient ring then $A(R)$ is a direct summand of R (i.e. $A(R)$ is generated by a central idempotent of R).

<u>Proof</u>

See 4.14 of [2]. □

§C. <u>Nilpotent radical, Invertible ideals</u>

The nilpotent radical of a ring R denoted by $N(R)$ or N is the sum of the nilpotent ideals of R (i.e. those ideals I with $I^n = 0$ for some integer $n > 0$).

<u>Proposition 1.7</u> (Lanski)

If R is a right Goldie ring then $N(R)$ is nilpotent.

<u>Proof</u>

This is Cor. 3.4 p. 27 of [6]. □

<u>Proposition 1.8</u>

If R is a semi-prime Goldie ring then R has only a finite number of minimal prime ideals. The intersection of these prime ideals is zero.

<u>Proof</u>

Lemma 1.16 of [2]. □

The above then tells us that if A is a semi-prime ideal of a ring R and R/A is a Goldie ring then A is the intersection of a finite number of prime ideals of R.

<u>Definition</u>

An ideal X of a ring R is said to be invertible if there exists an overing S of R s.t. if $X^{-1} = \{s \in S | sX \subseteq R \text{ and } Xs \subseteq R\}$ then $XX^{-1} = R = X^{-1}X$. If X contains a regular element and R has a quotient ring then S can always be taken to be the quotient ring of R. R is said to have enough invertible ideals if every ideal containing a regular element contains an invertible ideal containing a regular element.

An ideal I is said to be <u>AR</u> if for each right ideal K of R there exists $n > 0$ with $K \cap I^n \subseteq KI$ and similarly on the left.

§D. <u>Hereditary rings and projective modules</u>

We say a module P is projective if there exists a module K with $P \oplus K$ being isomorphic to a free module. A ring R is right hereditary (semi-hereditary) if every right ideal (f.g. right ideal) is projective. R is said to be *right bounded* if every essential right ideal contains an ideal which is essential as a right ideal of R. R is said to be *right fully bounded* if every prime factor ring of R is right bounded.

<u>Proposition 1.9</u>

Let R be a prime Noetherian hereditary ring. If I is an essential right ideal of R then $\frac{R}{I}$ is right Artinian. If $J(R) \neq 0$ then R is bounded, semi-local and J(R) is an invertible ideal of R and also R has enough invertible ideals.

Proof

The first part due to Chatters is 8.21 of [2]. For the rest see 4.13, 4.12 and 4.18 of [18]. □

Proposition 1.10 (A. Chatters)

If R is a Noetherian hereditary ring then R is the direct sum of prime rings and Artinian rings. Any prime ideal is either maximal or minimal.

Proof

This is 8.22 of [2]. The last part follows from the first part of 1.9. □

Proposition 1.11 (Dual Basis Lemma)

If M is a right R-module then M is projective iff it has a dual basis i.e. there are elements x_i of M and corresponding homomorphisms $f_i : M \to R$ such that if $x \in M$ then

(1) $f_i(x) = 0$ for all but finitely many i and

(2) $x = \sum_i x_i f_i(x)$.

Proof

This is Lemma 8.20 of [2]. □

It is then clear that every invertible ideal of a ring is right and left projective.

Proposition 1.12 (Schanuel's Lemma)

If we have two exact sequences of modules $0 \to K_1 \to P_1 \to B \to 0$ and $0 \to K_2 \to P_2 \to B \to 0$ where P_1 and P_2 are projective then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proof

See exercise 9 of §18 of [1]. □

§E. Localisation (Classical)

Definition

If R is a ring then a subset S of R is said to be *multiplicatively closed* iff $x \in S$, $y \in S \Rightarrow xy \in S$ and $1 \in S$.

We say a ring R_S is the right localisation of R. w.r.t. the multiplicatively closed set S iff there exists a ring homomorphism $\phi : R \to R_S$ such that

(1) $\phi(s)$ is invertible in R_S for $s \in S$.

(2) Every element in R_S is of the form $\phi(a)\,\phi(s)^{-1}$ for some $a \in R$, $s \in S$.

(3) $\phi(a) = 0 \Leftrightarrow as = 0$ for some $s \in S$.

If R_S exists it is unique up to isomorphism; we call R_S if it exists the *right localisation* of R w.r.t. S. If ${}_SR$ and R_S both exist then they are naturally isomorphic, if this occurs we say R_S is the *localisation* of R w.r.t. S. We refer the reader to [9] for more details.

To obtain necessary and sufficient conditions for the existence of R_S one introduces the following two conditions.

We say a multiplicatively closed set S is a *right Ore set* if given $s \in S$ and $b \in R$ there exists $t \in S$, $a \in R$ with $sa = bt$.

We say S is *right reversible* if $sa = 0$ with $s \in S$, $a \in R$ implies $at = 0$ for some $t \in S$. We then have:

Proposition 1.13

If S is a multiplicatively closed subset of a ring R then R is right localisable w.r.t. S iff S is right Ore and right Ore and right reversible. If R is Goldie then S right Ore $\Rightarrow$ S is right reversible.

Proof

This is 1.4 and 1.5 of [9]. □

One property of R_S we will use without comment is that given $q_1,\dots,q_n \in R_S$ there exists $s \in S$ such that $q_1 s,\dots,q_n s \in \phi(R)$. (See Ex. 1.2 of Chapter 2 of [9]). Intuitively one constructs R_S by first factoring out the torsion ideal $T = \{r \in R | rs = 0 \text{ for some } s \in S\}$ and then inverting the elements of S. Localisation w.r.t. Ore and reversible sets we will call *classical localisation.*

If I is an ideal of R we write $C_R(I)$ or $C(I) = \{x \in R | \bar{x} \text{ regular in } \frac{R}{I}\}$.

We say I is localisable if R is localisable w.r.t. C(I) and denote $R_{C(I)}$ by R_I. If $C_R(0) = C_R((0))$ is an Ore set the corresponding localisation $R_{C(0)}$ is called the *quotient ring* of R denoted by Q(R).

Proposition 1.14

Let X be a semi-prime ideal of the ring R then if $\frac{R}{X}$ is Goldie and X is localisable then $J(R_X) = XR_X$ and R_X is a semi-local ring.

Proof

See Theorem 3 of [10]. We give a direct proof for clarity. As $\frac{R}{X}$ is Goldie $X = P_1 \cap \ldots \cap P_n$ where the P_i are the minimal primes over X (1.8). We can assume $C(X) \subseteq C(0)$ so $R \hookrightarrow R_X$. $C(X) = C(P_1) \cap \ldots \cap C(P_n)$ so then $R_X P_i R_X \cap R = P_i$ and $C(X) \subseteq C(P_i)$; so $P_i R_X = R_X P_i R_X = R_X P_i$. Also $Q(\frac{R}{X}) \cong \frac{R_X}{XR_X}$ and is therefore Artinian by Goldie's theorem. If M is a maximal right ideal of R_X then if $M \not\supseteq XR_X$ then $M + XR_X = R_X$ and so $M \cap C(X) \neq \emptyset$ so $M = R_X$. Thus $M \supseteq XR_X$ and we see $XR_X = J(R_X)$ and R_X is semi-local. □

Proposition 1.15 (Faith)

Suppose R has a quotient ring Q then Q is semi-local iff there exists a semi-prime ideal $H = P_1 \cap \ldots \cap P_n$ with $C(H) = C(0)$ and $\frac{R}{H}$ is a Goldie ring. If this holds $C(H) = C(P_1) \cap \ldots \cap C(P_n)$, $P_i Q = QP_i$, $i = 1,\ldots,n$ are the maximal ideals of Q and $P_i Q \cap R = P_i$. The $\frac{R}{P_i}$ are Goldie.

Proof

See either Theorem (D) of [20] or Cor. 4.4 of [10]. □

§F. Orders

A ring R is said to be an order if it has a quotient ring Q, and we say R is an order in Q.

If A,B are subsets of Q we then define:

$O_r(A) = \{x \in Q \mid Ax \subseteq A\}$ - the right order of A.

$O_\ell(A) = \{x \in Q \mid xA \subseteq A\}$ - the left order of A.

$(A:B)_r = \{x \in Q \mid Bx \subseteq A\}$, $(A:B)_\ell = \{x \in Q \mid xB \subseteq A\}$.

$(A:B)_R = \{x \in R \mid Bx \subseteq A\}$, ${}_R(A:B) = \{x \in R \mid xB \subseteq A\}$.

$A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.

A right R-module A contained in Q is called a *right R-ideal* if it satisfies the following:

(1) There exists $c \in C(0)$ with $cA \subseteq R$.

(2) $A \cap C_R(0) \neq \emptyset$.

Similarly for left R-ideals. A right and left R-ideal is called an R-ideal.

If $(R:A)_r = (R:A)_\ell$ we will sometimes write A^* for $(R:A)_r$ as it is more compact.

If A is a left or right R-ideal we write

$$A_\nu = (R:(R:A)_\ell)_r, \quad {}_\nu A = (R:(R:A)_r)_\ell.$$

A right R-ideal with $A = A_\nu$ is called a *right ν-ideal* or just *right reflexive* (similarly on left). A set A contained in Q is called *integral* if $A \subseteq R$. For compactness the term 'reflexive ideal' (or 'ν-invertible ideal' see below for definition) will mean an integral reflexive (ν-invertible) R-ideal. If R has no reflexive ideals then we say R is ν-*simple*. The term 'of R' in a sentence means integral. A right (left) R-ideal of R is *regular* if it contains an R-ideal. R is said to be *regular* if every one-sided R-ideal of R is regular.

Let R and T be two orders in Q then we say R and T are left (right) equivalent if there exists units a, b in Q with $Ra \subseteq T$ ($aR \subseteq T$) and $Tb \subseteq R$ ($bT \subseteq R$). R and T are equivalent iff there exist units a, b, c, d in Q with $aRb \subseteq T$ and $cTd \subseteq R$.

If R is an order in Q we say R is a *maximal order* if R satisfies the equivalent conditions in the following proposition.

Proposition 1.16

If R is an order in Q then the following are equivalent.

(1) For all left R-ideals A, $O_\ell(A) = R$ and for all right R-ideals B one has $O_r(B) = R$.

(2) Same as (1) but assume A and B are integral.

(3) For all R-ideals I one has $O_\ell(I) = O_r(I) = R$.

(4) Same as (3) but assume I is integral.

(5) There exists no order in Q equivalent to R and strictly containing R.

(6) Same as (5) but replace equivalent by right or left equivalent.

Proof

See 3.1 and 2.3 of [7]. □

An order is called Asano if every integral R-ideal is invertible. A maximal order such that every one-sided R-ideal is projective is called a *Dedekind order.* A Dedekind order is Asano.

Proposition 1.17

If R is a maximal order in Q then we have the following:

(1) If $Q = Q_1 \oplus \ldots \oplus Q_n$ then $R = R_1 \oplus \ldots \oplus R_n$ where $Q(R_i) = Q_i$.

(2) If I is a reflexive right R-ideal then $O_\ell(I)$ is a maximal order equivalent to R.

Proof

For (1) see 5.2 of [33] and for (2) see Chapter 1, 3.2 of [7]. □

Proposition 1.18

Let R be an order in Q. Then if I is a right R-ideal than $O_\ell(I) = II^{-1}$ iff I is a projective right $O_r(I)$-ideal and then I is a f.g. right $O_r(I)$-ideal. If I is a projective right R-ideal then I_R is f.g. and $II^{-1} = I(R:I)_\ell = O_\ell(I)$.

Proof

See 1.2 and 1.4 of [33]. □

Proposition 1.19

Let R be an order in Q. If A and B are two right R-ideals then $(A:B)_\ell \cong \mathrm{Hom}_R(B_R, A_R)$ as abelian groups.

Proof

See Chapter 3, 1.1 of [7]. □

Another class of orders which have a 'good' multiplicative structure of ideals are hereditary Noetherian prime rings with enough invertible ideals. In [21] and [30] a generalisation of these rings were introduced; these we define below but we do not assume that R is prime. Also our definition will be slightly different but will agree with Fujita's definition for additive regular rings (see Chapter 2).

A ν-ideal X is said to be *ν-invertible* if $O_\ell(X) = R = O_r(X)$. A ν-ideal A is said to be *ν-idempotent* if $(A^2)_\nu = {}_\nu(A^2) = A$. R is said to have *enough ν-invertible ideals* if every integral R-ideal which is right (left) reflexive contains a ν-invertible ideal. A right R-ideal A is said to be *right ν-projective* if ${}_\nu(A(R:A)_\ell) = {}_\nu(O_\ell(A))$. A left R-ideal B is said to be *left ν-projective* if $((R:B)_r B)_\nu = (O_r(B))_\nu$.

We say R is a *ν-Hereditary order* or a νH-*order* if for every integral R-ideal A which is left (right) reflexive A is right (left) ν-projective.

i.e. $A = {}_\nu A \Rightarrow {}_\nu(A(R:A)_\ell) = {}_\nu(O_\ell(A))$

and $A = A_\nu \Rightarrow ((R:A)_r A)_\nu = (O_r(A))_\nu$.

The natural domain of νH-orders are additive regular orders as for these rings $A = {}_\nu A \Rightarrow {}_\nu(O_\ell(A)) = O_\ell(A)$ (A an R-ideal), see Chapter two.

We will call a νH-order with enough ν-invertibles an e-νH-order.

CHAPTER TWO - ADDITIVE REGULAR RINGS AND e-νH-ORDERS.

§A. Additive regular rings

Definition

We say a ring is additive regular iff for any right (left) ideal I of R with $I \cap C(0) \neq \emptyset$ and $a \in R$ we have $(a+I) \cap C(0) \neq \emptyset$.

The reason why these rings are introduced will be obvious from what follows. They are useful in that they enable us to produce 'partial' localisations corresponding to Ore sets which need not be right reversible.

By 1.2 semi-prime Goldie rings are additive regular, also in 2.3 we will see Noetherian orders and rings with semi-local quotient rings are additive regular.

Proposition 2.1

If R is an additive regular order then

(1) Any right R-ideal is generated by it s regular elements.

(2) Any overing R' of R contained in the quotient ring of R is additive regular.

Proof

(1) Is obvious.

(2) Suppose I is a right R'-ideal of R'. Let $u \in I \cap C_{R'}(0)$ and $a \in R'$. There exists $e \in C_R(0)$ s.t. $ea \in R$ and $eu \in R$, so $eu \in C_R(0)$ so $(ea + euR) \cap C_R(0) \neq \emptyset$ and hence $(a + uR') \cap C_{R'}(0) \neq \emptyset$. $\square$

Proposition 2.2

If R has a semi-local quotient ring then R is additive regular.

Proof

By 1.15 there exists a semi-prime ideal H with $\frac{R}{H}$ Goldie and $C(0) = C(H)$. Let I be a right ideal of R with $I \cap C(0) \neq \emptyset$ and $a \in R$ then $(\bar{a} + \bar{I}) \cap \bar{R}$. contains a regular element of $\bar{R} = \frac{R}{H}$ so $(a + I) \cap C(0) \neq \emptyset$. □

Theorem 2.3 (Stafford)

Any Noetherian order has a semi-local quotient ring.

Proof

Corollary 5.4 of [37]. □

The above theorem which had been a conjecture until recently is crucial to the study of non-prime orders. The following theorem was proved for Noetherian rings in Artinian quotient rings in [27]. By using 2.3 we can give a simple proof.

Theorem 2.4

Suppose R is an additive regular order and A is an ideal of R with $A \cap C(0) \neq \emptyset$ then

(1) If K is a right ideal of R such that $K \cap C(A) \neq \emptyset$ and $K \cap C(0) \neq \emptyset$ then $K \cap (C(A) \cap C(0)) \neq \emptyset$.

(2) If $C(A)$ is an Ore set then $C(A) \cap C(0)$ is an Ore set.

Proof

(1) Let $a \in C(A) \cap K$ then $a + KA$ contains a regular element c, then $c \in K \cap (C(A) \cap C(0))$.

(2) Let $c \in C(A) \cap C(0)$, $r \in R$, there exists $r_1, r_2 \in R, e_1 \in C(A)$,

$e_2 \in C(0)$ with $cr_1 = re_1$, $cr_2 = re_2$. Let $K = e_1R + e_2R$ then by (1) $K \cap (C(A) \cap C(0)) \neq \emptyset$ so there exists $f \in C(A) \cap C(0)$ with $f = e_1s_1 + e_2s_2$ for some $s_1, s_2 \in R$ so $c(r_1s_1 + r_2s_2) = rf$. It follows that $C(A) \cap C(0)$ is an Ore set. □

Related to the above is the following.

<u>Lemma 2.5</u>

If R is a ring and $P_1, \ldots, P_n$ are prime ideals with R/P_i Goldie then if K is a right ideal of R and $K \cap C(P_i) \neq \emptyset$ for each i then $K \cap (C(P_1) \cap \ldots \cap C(P_n)) \neq \emptyset$.

<u>Proof</u>

One notes that in Lemma 13.4 of [2] one only needs the above hypothesis rather than the Noetherian hypotheses given there. □

The lemmas below are needed to generalise Section one of [30].

<u>Lemma 2.6</u>

Let R be an additive regular order. If I is a right R-ideal then I_ν equals the intersection of principal right R-ideals containing I.

<u>Proof</u>

This is well known for prime Goldie rings. Let I be a right R-ideal then clearly $I_\nu \subseteq \cap tR$ where the intersection is over the principal right R-ideals containing I. Also if $u \notin I_\nu$ then since $(R:I)_\ell$ is generated by it s regular elements, there exists $t \in (R:I_\nu)_\ell \cap C_{Q(R)}(0)$ with $tu \notin R$, so $u \notin t^{-1}R$ a principal right R-ideal containing I_ν. □

Corollary 2.7

If R is an additive regular order in Q then if I is a right R-ideal and J a right R-ideal with $RJ \subseteq J$ then $(IJ)_\nu = (IJ_\nu)_\nu$.

Proof

We have $(IJ)_\nu \subseteq (IJ_\nu)_\nu$, now IJ is a right R-ideal as $RJ \subseteq J$ so $(IJ)_\nu$ is a right R-ideal. If $(IJ)_\nu \subseteq cR$, c a unit of Q then $IJ \subseteq cR$ so $c^{-1}IJ \subseteq R$ so $c^{-1}IJ_\nu \subseteq R$ so $(IJ_\nu)_\nu \subseteq cR$ so by Lemma 2.6 $(IJ)_\nu = (IJ_\nu)_\nu$. □

§B. e-νH-orders

We wish to generalise Section one of [30] to additive regular e-νH-orders. This follows easily from Lemma 2.6 and Corollary 2.7. As Marubayashi's proof involves references to unpublished work we will give an outline of the proofs. Let us first show our definition of νH-orders coincides with that of [30] in the case of additive regular e-νH-orders. We first note that if A is an R-ideal with $A = A_\nu$ then $(O_r(A))_\nu = O_r(A)$. This is true as $A(O_r(A))_\nu \subseteq (AO_r(A))_\nu \subseteq A$ by 2.7.

Proposition 2.8

Let R be an additive regular e-νH-order then:

(1) If A is an integral R-ideal and $A_\nu = A$ then ${}_\nu A = A$.

(2) If A is an R-ideal with $A = A_\nu$ then A is left ν-projective.

Proof

(1) Let A be an integral R-ideal with $A = A_\nu$, now ${}_\nu A$ is a left R-ideal so is generated by it s regular elements by 2.1. Let $q \in {}_\nu A$ and q regular

so $q(R:A)_r \subseteq R$ so $q(R:A)_r A \subseteq A$, so $q \in qO_r(A) = q((R:A)_r A)_\nu = (q(R:A)_r A)_\nu \subseteq A_\nu = A$.

(2) There is a ν-invertible ideal X of R s.t. $XA \subseteq R$ and so by (1) $(XA)_\nu = {}_\nu(XA)$.

Now we have $(R:A)_r \supseteq (R:XA)_r X$ for the R.H.S. is generated by it's regular elements so let q be regular and $q \in$ R.H.S. then $qX^{-1} \subseteq (R:XA)_r$ so $XAqX^{-1} \subseteq R \Rightarrow X^{-1}XAq \subseteq R$ so ${}_\nu(X^{-1}X)Aq \subseteq R$ so $Aq \subseteq R$ so $q \in (R:A)_r$. It is also clearly true that $O_r((XA)_\nu) \supseteq O_r(A)$.

Therefore we have $((R:A)_r A)_\nu \supseteq ((R:(XA)_\nu)_r XA)_\nu = ((R:(XA)_\nu)_r (XA)_\nu)_\nu = O_r((XA)_\nu) \supseteq O_r(A) \supseteq ((R:A)_r A)_\nu$ so $((R:A)_r A)_\nu = O_r(A)$ and so A is left ν-projective. □

(2) of 2.8 shows us our definition of νH-orders coincides with that of [30] in the case of additive regular e-νH-orders. Also we can repeat (1) for any R-ideal A so $A_\nu = {}_\nu A$ for all R-ideals A in an additive regular e-νH-order.

<u>Proposition 2.9</u>

Let R be an additive regular νH-order then the following are equivalent for a ν-ideal A of R.

(1) A is ν-invertible.

(2) $(A(R:A)_r)_\nu = R = {}_\nu((R:A)_\ell A)$.

(3) $(R:A)_r = A^{-1} = (R:A)_\ell$.

<u>Proof</u>

(1) $\Rightarrow$ (3) follows from the definition. (3) $\Rightarrow$ (2) and (1), for if (3) holds then $A(R:A)_\ell \subseteq R$ so ${}_\nu(A(R:A)_\ell) \subseteq R \subseteq O_\ell(A) \subseteq {}_\nu(A(R:A)_\ell)$. (1) $\Rightarrow$ (2)

for if $qA(R:A)_r \subseteq R$ then $qA \subseteq A \Rightarrow q \in R$. (2) $\Rightarrow$ (1) for if $qA \subseteq A$ then $qA(R:A)_r \subseteq R$ so $q(A(R:A)_r)_\nu \subseteq R$ so $q \in R$. □

<u>Lemma 2.10</u>

If R is an additive regular e-νH-order and if A is a ν-ideal of R s.t. $A = (A(R:A)_r)_\nu$ then A is ν-idempotent.

<u>Proof</u>

$$(A^2)_\nu = ((A(R:A)_r)_\nu A)_\nu \supseteq (A((R:A)_r A)_\nu)_\nu = (AO_r(A))_\nu = A_\nu = A. \quad \square$$

<u>Proposition 2.11</u>

If R is an additive regular e-νH-order and if M is a maximal ν-ideal of R then M is a prime ideal, any such ideal is ν-invertible or ν-idempotent.

<u>Proof</u>

The first part is clear, for the second part note that if M is not ν-invertible then by 2.9 either $(M(R:M)_r)_\nu \subsetneq R$ or ${}_\nu((R:M)_\ell M) \subsetneq R$ so one of these terms equals M and hence M is ν-idempotent by 2.10. □

<u>Definition</u>

A set of distinct maximal ν-ideals of R which are ν-idempotent $M_1,\ldots,M_n$ s.t. $O_r(M_1) = O_\ell(M_2),\ldots,O_r(M_n) = O_\ell(M_1)$ is called a *cycle*. We also regard a maximal ν-invertible ideal as a cycle.

Proposition 2.12

If R is an additive regular e-νH-order then any two cycles of R either coincide or are disjoint.

Proof

As ν-invertible ideals cannot be ν-idempotent we only need consider cycles of ν-idempotents but if $O_\ell(M) = O_\ell(N)$ for M,N ν-idempotent ideals of R then $M = (R:(R:M)_\ell)_r = (R:(R:N)_\ell)_r = N$. □

Lemma 2.13

If R is an additive regular e-νH-order and X is any ν-invertible ideal of R then there is a 1-1 correspondence between ν-idempotents of R containing X and the ν-ideals W s.t. W is a ring and $R \subseteq W \subseteq X^{-1}$ which is given by $A \to O_\ell(A) = (R:A)_\ell$, $W \to (R:W)_r$.

Similarly there is a 1-1 correspondence given by $A \to O_r(A)$ and $W \to (R:W)_\ell$.

Lemma 2.14

If R is an additive regular e-νH-order satisfying the maximum condition on ν-ideals of R, then any descending chain of ν-ideals containing a fixed ν-invertible ideal of R stops.

The proof of the last two lemmas is analogous to 1.6 and 1.7 of [30] using Lemma 2.10 above. □

Proposition 2.15

If R is an additive regular e-νH-order satisfying the a.c.c on

ν-ideals of R and X is a ν-invertible ideal of R and M is a ν-idempotent maximal ν-ideal of R containing X then there is a cycle $M_1,\dots,M_n$, with $M = M_1$ and $\bigcap_1^n M_i \supseteq X$.

Proof

We first show that there is a ν-idempotent maximal ν-ideal of R M' say such that $O_r(M) = O_\ell(M')$ and $M' \supseteq X$. Let $W = O_r(M)$, $M' = (W:R)_r$, then $M' \supseteq X$, M' is a ν-ideal and is a maximal ν-idempotent R ideal of R by 2.13. If M' is not a maximal ν-ideal then there exists P, P a ν-invertible ideal containing M'. So $M' = (M'^2)_\nu \subseteq (P^2)_\nu$, similarly $X \subseteq M' \subseteq (P^n)_\nu$ for all n; so by 2.14 $(P^n)_\nu = (P^{n+1})_\nu$ some $n \geq 1$ so P = R so M' is a maximal ν-ideal. Repeating this process we get $M_1,\dots,M_r,\dots$ each M_i being a ν-idempotent maximal ν-ideal containing X with $M = M_1$. We have $M_1 \supseteq M_1 \cap M_2 \supseteq \dots \supseteq X$. As $\bigcap_1^r M_i$ is a ν-ideal by 2.14 the chain stops so $M_i = M_{n+i}$ for some $n > 1$ and $i \geq 1$; choose the smallest such i then if $i > 1$ we have $O_r(M_{n+i-1}) = O_\ell(M_{n+i}) = O_\ell(M_i) = O_r(M_{i-1})$ so i = 1 and we obtain our required cycle. □

Proposition 2.16

Let R be an additive regular e-νH-order with the maximum condition on ν-ideals. Let $M_1,\dots,M_n$ be a union of cycles of R then $X = \bigcap_1^n M_i$ is a ν-invertible ideal of R.

Proof

Suppose X is not ν-invertible then either $(X(R:X)_r)_\nu \subsetneq R$ or $_\nu((R:X)_\ell X) \subsetneq R$. Suppose $(X(R:X)_r)_\nu \subsetneq R$ then there exists a maximal ν-ideal

M of R such that $M \supseteq (X(R:X)_r)_\nu \supseteq X \supseteq M_1 \dots M_n$ so $M = M_j$ for some j. First suppose M_j is ν-idempotent then $n > 1$ and we can assume $j > 1$, $O_r(M_{j-1}) = O_\ell(M_j)$. Let $A = M_1 \dots M_{j-2}M_{j+1} \dots M_n$.

Then $M_j \supseteq (X(R:X)_r)_\nu \supseteq (X(R:M_{j-1})_r)_\nu \supseteq (AM_{j-1}M_j(R:M_{j-1})_r)_\nu \supseteq AM_{j-1}(M_j(R:M_{j-1})_r)_\nu = AM_{j-1}(M_jO_r(M_{j-1}))_\nu = AM_{j-1}(M_jO_\ell(M_j))_\nu = AM_{j-1}(M_j(R:M_j)_\ell)_\nu = AM_{j-1}O_\ell(M_j) = AM_{j-1}$ so $M_j = M_k$ for some $k \neq j$ a contradiction. Otherwise M_j is ν-invertible but then we put $B = M_1 \dots M_{j-1}M_{j+1} \dots M_n$ then $M_j \supseteq (X(R:X)_r)_\nu \supseteq (X(R:M_j)_r)_\nu = (XM_j^{-1})_\nu \supseteq (BM_jM_j^{-1})_\nu = B_\nu \supseteq B$ so $M_j = M_i$, $j \neq i$ so again we obtain a contradiction and we conclude X must be ν-invertible. □

<u>Proposition 2.17</u>

Let R be an additive regular e-νH-order satisfying the maximum condition on ν-ideals of R then if P is an R-ideal of R then P is a maximal ν-invertible ideal iff it is the intersection of a cycle.

<u>Proof</u>

From above we only have to prove the intersection of a cycle is a maximal ν-invertible ideal. If P is prime then this is obvious, else $P = M_1 \cap \dots \cap M_n$, where $M_1, \dots, M_n$ is a cycle. Suppose P is not ν-invertible then $P \subseteq P' \subsetneq R$ for some P' a maximal ν-invertible ideal, write $P' = N_1 \cap \dots \cap N_\ell$ then $M_i = N_j$ for some i,j and so $P = P'$ by 2.12. □

Proposition 2.18

If R is an additive regular e-νH-order satisfying the maximum condition on ν-ideals of R then $D(R) = \{X | X \text{ a } \nu\text{-invertible R-ideal}\}$ is an abelian group under the multiplication $A \circ B = (AB)_\nu = {}_\nu(AB)$.

Proof

Clear but note D(R) is abelian for if P_1, P_2 are two maximal ν-invertible ideals of R then $P_1 \cap P_2$ is ν-invertible being the intersection of cycles and we see $P_1 \circ P_2 = P_2 \circ P_1 = P_1 \cap P_2$. $\square$

CHAPTER THREE - LOCALISATION

§A. General Localisation

Let R be a ring, a set of right ideals F of R is called a (right) Gabriel topology if it satisfies the following.

(T1) If $A \in F$ and B is a right ideal containing A then $B \in F$.

(T2) If $A \in F$ and $B \in F$ then $A \cap B \in F$.

(T3) If $A \in F$ and $x \in R$ then $(A:x)_R \in F$.

(I) If A is a right ideal and if there exists $B \in F$ s.t. for all $b \in B$ $(A:b)_R \in F$ then $A \in F$.

If F is a Gabriel topology then it is closed under products. If σ is a functor from Mod-R to Mod-R then we say σ is an idempotent kernel functor if for every $M \in$ Mod-R $\sigma(M) \subseteq M$ and if $f \in \mathrm{Hom}_R(M,N)$ then $f(\sigma(M)) \subseteq \sigma(N)$. Also if $N \subseteq M$ then $\sigma(N) = N \cap \sigma(M)$ and $\sigma(\frac{M}{\sigma(M)}) = 0$.

We have a 1-1 correspondence between the idempotent kernel functors and Gabriel topologies namely: Given F define $\sigma(M) = \{x \in M | xA = 0$ for some $A \in F\}$ and given σ let F be defined by $A \in F$ iff $\sigma(R/A) = R/A$. For a proof of this see §5 of Chapter 5 of [9]. For a simple exposition of General localisation see [8].

Given a right Gabriel topology F with corresponding idempotent kernal functor σ we say a module M is F-*torsion* if $\sigma(M) = \{x \in M | xA = 0$ for some $A \in F\} = M$ and F-*torsion free* if $\sigma(M) = 0$.

All Gabriel topologies can be cogenerated by an injective module i.e. there exists an injective module E such that $F = \{I | I$ a right ideal with $\mathrm{Hom}(\frac{R}{I}, E) = 0\}$. Given a right Gabriel topology F we construct the localisation of a module M w.r.t. F as follows: Form $\sigma(M) = \{m \in M | mF = 0$ for some $F \in F\}$. Then define $M_F = \{x \in E_R(\frac{M}{\sigma(M)}) |$ there exists $I \in F$ with $xI \subseteq \frac{M}{\sigma(M)}\}$, where $E_R(\_)$ denotes the injective hull.

When $M = R$, R_F is a ring called the localisation of R with respect to F. If M is a module then M_F is a right R_F-module. If I is a right ideal of R we write

$$Cl_r(F)(I) = \{r \in R | rH \subseteq I \text{ for some } H \in F\}.$$

If $I = Cl_r(F)(I)$ we say I is F-closed and if R satisfies the maximum condition on F-closed right ideals we say R is *F-Noetherian*.

<u>Proposition 3.1</u>

Let R be an order and F a (right) Gabriel topology then $F_o = \{I \in F | I \cap C(0) \neq \emptyset\}$ is a right Gabriel topology.

<u>Proof</u>

(T_1), (T_2), (T_3) are clear using the Ore condition; for (I) merely note that if A is a right ideal and $B \in F_o$ with $(A:b)_R \subseteq F_o$ for all $b \in B$ then $A \in F$ and there exists $b \in B$, b regular and $(A:b)_R \in F_o$ and so $A \in F_o$. □

§B. <u>r-τ-Noetherian and τ-Noetherian orders</u>

One of the most important properties of prime maximal orders is that the reflexive ideals 'tend' to be localisable. This was proved for prime Noetherian maximal orders by Chamarie in [15]. However, the

Noetherian condition is theoretically too strong for maximal orders. This is because orders equivalent to Noetherian maximal orders may not be Noetherian although such an order is contained in a maximal order equivalent to R. Also general localisation does not tend to preserve the Noetherian condition but preserves the maximal order property. In the important paper [14] Chamarie overcame these problems by introducing the Gabriel topology τ cogenerated by $E_R(\frac{Q}{R})$ calling prime Goldie τ-Noetherian orders Krull orders, these are the natural non-commutative generalisation of Krull domains. The reflexive ideals are still localisable. We are interested in the non-prime situation. Here we see that there are two natural chain conditions which we define below, one giving reflexive ideals localisable the other $C(A) \cap C(0)$ localisable.

Proposition 3.2

Let R be an order in Q then if I is a right ideal of R then the following are equivalent.

(1) For all $a \in R$ $(R:(I:a)_R)_\ell = R$.

(2) $\mathrm{Hom}_R(\frac{R}{I}, E(\frac{Q}{R})) = 0$.

Proof

See Prop. 1 of [28]. □

We denote the set of right ideals of R satisfying 3.2 by τ_r, similarly for left hand side.

We say R is *τ-Noetherian* if it is τ_r-Noetherian and τ_ℓ -Noetherian. R is r-τ-Noetherian if R satisfies the maximum condition on the closed right (left) R-ideals of R. Similarly for τ_o-Noetherian where

$\tau_{or} = \{I \in \tau_r | I \cap C(0) \neq \emptyset\}$. We have the implications:

$$\begin{array}{ccc} \tau_o\text{-Noetherian} & \longrightarrow & \tau\text{-Noetherian} \\ \downarrow & & \downarrow \\ r\text{-}\tau_o\text{-Noetherian} & = & r\text{-}\tau\text{-Noetherian}. \end{array}$$

These chain conditions all coincide when R is prime Goldie. In the same vein if P is a property of right (left) ideals of a ring then we say a ring R is a r-P-*ring* if the right (left) R-ideals of R satisfy P.

We say R is a *Krull order* if it is an r-τ-Noetherian maximal order. A τ-Noetherian maximal order we call a *strong Krull order*. We will write $Cl_r(I)$ or just $Cl(I)$ for $Cl(\tau_r)(I)$. We note that if I is a right R-ideal of R then $I \subseteq Cl(I) \subseteq I_\nu$ for suppose $x \in Cl(I)$ then there exists $H \in \mathcal{T}_r$ with $xH \subseteq I$ so $(R:I)_\ell xH \subseteq R$ so $(R:I)_\ell x \subseteq R$ so $x \in I_\nu$.

§C. Ore sets

Like Marubayashi in [30] we wish to generalise the work of Section 1 of [14] but we are concerned with $C(A) \cap C(0)$ as well as $C(0)$ being an Ore set.

Proposition 3.3

Let R be an additive regular r-τ-Noetherian e-νH-order, then if A and B are ν-invertible ideals of R then $C(A \circ B) = C(A) \cap C(B)$.

Proof

Let $c \in C(A \circ B)$, suppose $cx \in A$ then $c(x + A) \subseteq A$, as $(x + A) \cap C(0) \neq \emptyset$ $cy \in A$ for some $y \in (x + A) \cap C(0)$, so $cyB \subseteq A \circ B$ so

$yB \subseteq A \circ B$ so $(RyBB^{-1})_\nu \subseteq A$ so $y \in A$ so $x \in A$, so $C(A \circ B) \subseteq C(A) \cap C(B)$ the converse is similar. □

Proposition 3.4 .

Let R be an r-τ-Noetherian additive regular e-νH-order, then if I is an R-ideal of R which is right closed then $\frac{R}{I}$ is right Goldie. If R is τ-Noetherian we only need I to be a closed right R-ideal.

Proof

Let $K = r(\bar{S})$ be a right annihilator ideal in R/I, where S is a subset of R, i.e. $s \in S \Rightarrow sK \subseteq I$. Then if $q \in Cl(K)$ so there exists $F \in \tau_r$ with $qF \subseteq K$ then $sqF \subseteq I$ so $sq \in I$ so $q \in K$ i.e. K is closed, also $I \subseteq K$ as $RI \subseteq I$ so R satisfies the maximum condition on right annihilators of subsets of $\frac{R}{I}$. Now suppose $L \cap K \subseteq I$, $L, K \supseteq I$, L, K right ideals of R then $K \cap Cl(L) \subseteq I$ for $x \in K \cap Cl(L) \Rightarrow xF \subseteq L$ for some $F \in \tau_r$ so $xF \subseteq L \cap K \subseteq I$ so $x \in I$. Thus, if we have a direct sum of right R-ideals $I_1, \ldots, I_n, \ldots$ mod I then there exists $r > 1$ with $Cl_r(\sum_1^r I_i) = Cl_r(\sum_1^{r+1} I_i)$ so $I_{r+1} \cap Cl_r(\sum_{i=1}^{r+1} I_i) \subseteq I$ so $I_{r+1} \subseteq I$. This contradiction shows $\frac{R}{I}$ has finite uniform dimension. So we have proved $\frac{R}{I}$ is right Goldie. When R is τ-Noetherian we don't require in the first part of the proof that K be a right R-ideal so we don't need the condition $RI \subseteq I$. □

Proposition 3.5 (Hajarnavis, Small)

Let R be a ring. Then R has a right Artinian right quotient ring iff R satisfies the following conditions.

(1) R and $\frac{R}{N}$ are right Goldie rings.

(2) $\frac{R}{\ell(N^k)}$ is a right finite-dimensional ring for $k = 1,\dots,p-1$ where p is an integer such that $N^p = 0$ but $N^{p-1} \neq 0$.

(3) $C(N) \subseteq C(0)$ in R.

Proof

Theorem 2.10 of [25]. □

Proposition 3.6

Let R be an additive regular r-τ-Noetherian e-νH-order. If A is a ν-invertible ideal then $\frac{R}{A}$ has an Artinian quotient ring.

Proof

We have $A = P_1^{n_1} \circ \dots \circ P_h^{n_h}$ where the P_i are the maximal ν-invertible R-ideals of R containing A and the $n_i > 0$. If $N = \bigcap_1^h P_i$ and $N' = \frac{N}{A}$ then N' is the nilpotent radical of $\frac{R}{A} = R'$ and $C(A) = C(N)$. $\frac{R}{N}$ is Goldie as N is reflexive and hence closed. So by 3.5 we need only show $\frac{R'}{\ell(N'^m)}$ is right finite-dimensional. Let $I_m = \{r \in R \mid rN^m \subseteq A\}$. By 3.4 we need only show I_m is right closed, so suppose $qF \subseteq I_m$ for some $q \in R$ and $F \in \tau_r$ then as $I_m \cap C(0) \neq \emptyset$ we can suppose $F \cap C(0) \neq \emptyset$ and that q is regular. Now $qFN^m \subseteq A$ so $qF(N^m)_\nu \subseteq A$ but $(N^m)_\nu$ is ν-invertible so $qF \subseteq (AN^{-m})_\nu = (N^{-m}A)_\nu$. Thus $N^m qF \subseteq A$ so $N^m q \subseteq A$ so $qN^m \subseteq A$ and $q \in I_m$. Thus I_m is right closed. Thus $\frac{R}{A}$ has an Artinian quotient ring. □

We will later show that if R has a semi-local quotient ring then $\frac{R}{A}$ has a generalised uniserial quotient ring.

Proposition 3.7

Let R be an additive regular r-τ-Noetherian e-νH-order then if A is a ν-invertible R-ideal of R and $F \in \tau_r$ then $F \cap C(A) \neq \emptyset$.

Proof

From above we can assume that A is a semi-prime ideal and as $\frac{R}{A}$ is Goldie we need only show any $F \in \tau_r$ is essential in $\frac{R}{A}$. Suppose $F \cap I \subseteq A$ for some right ideal I of R then if $x \in I$ we have $x(F:x)_R \subseteq A$ so $x \in A$ and $I \subseteq A$. Hence F is essential in $\frac{R}{A}$ so $F \cap C(A) \neq \emptyset$. □

Proposition 3.8

If R is an additive regular e-νH-order, A a ν-invertible R-ideal of R and I a right ideal of R then we have $Cl_r(I)A \subseteq Cl_r(IA)$.

Proof

If $x \in Cl_r(I)$ and $a \in A$ then there exists $H \in \tau_r$ with $xH \subseteq I$ so $xa(HA:a)_R \subseteq IA$. We therefore only need show that $(HA:a)_R \in \tau_r$ for all $a \in A$. Now $((HA:a)_R:r)_R = (HA:ar)_R$ for all $r \in R$, so we need only show $(R:(HA:a)_R)_\ell = R$ for all $a \in A$. Suppose $q(HA:a)_R \subseteq R$, let $t \subseteq A^{-1}$ then $qt(H:at)_R A \subseteq q(H:a)_R A \subseteq q(HA:a)_R \subseteq R$, so $Aqt(H:at)_R \subseteq R$ and so $Aqt \subseteq R$ this is true for all $t \in A^{-1}$ so $AqA^{-1} \subseteq R$ and so $Aq \subseteq A$, so $q \in R$. □

Proposition 3.9

Let R be an additive regular r-τ-Noetherian e-νH-order, then if I is a right ideal of R with $I \cap C(0) \neq \emptyset$ then there exists $n > 0$ with $I \cap (A^n)_\nu \subseteq Gl_r(IA)$. If we assume R is τ-Noetherian then we do not need to assume $I \cap C(0) \neq \emptyset$. (A is a ν-invertible ideal).

Proof

There exists $n > 0$ with $Cl_r(\sum_{h=1}^{\infty} (I \cap (A^h)_\nu)(A^{-h})_\nu) =$ $Cl_r(\sum_1^{n-1} (I \cap (A^h)_\nu)(A^{-h})_\nu)$. Therefore $(I \cap (A^n)_\nu)(A^{-n})_\nu \subseteq$ $Cl_r(\sum_1^{n-1} (I \cap (A^h)_\nu)(A^{-h})_\nu)$ so $(I \cap (A^n)_\nu)(A^{-n})_\nu(A^n)_\nu \subseteq$ $Cl_r(\sum_1^{n-1} (I \cap (A^h)_\nu)(A^{-h})_\nu)(A^n)_\nu \subseteq Cl_r(IA)$ by 3.8. Similarly for the last part. □

Theorem 3.10

If R is an additive regular e-νH-order and A is a ν-invertible ideal of R then

(1) If R is r-τ-Noetherian then $C(A) \cap C(0)$ is an Ore set.

(2) If R is τ-Noetherian then $C(A)$ is an Ore set.

Proof

(1) Let $c \in C(A) \cap C(0)$, $x \in R$. We have $C(A) = C((A^h)_\nu)$ by 3.3 for all $h \geq 1$. By 3.6 $\frac{R}{(A^h)_\nu}$ has a quotient ring.

Therefore for each $h \geq 1$ there exists $z_h, y_h, d_h \in R$ with $d_h \in C(A)$ s.t. $z_h = cy_h - xd_h \in (A^h)_\nu$. Let $I = \sum_1^\infty z_h R + cR$ so $I \cap C(0) \neq \emptyset$. Then there exists $n \geq 1$ s.t. $I \cap (A^n)_\nu \subseteq Cl_r(IA)$ by 3.9. So we have $z_n \in Cl_r(IA) \subseteq Cl_r(cA + \sum_1^\infty z_h A)$. There exists $F \in \tau_r$ with $z_n F \subseteq cA + \sum_1^\infty z_h A$. $F \cap C(A) \neq \emptyset$ by 3.7. So there exists $d \in C(A)$ with $z_n d = ca + \sum_{h=1}^m z_h a_h$ for some a_h, $a \in A$, then $(cy_n - xd_n)d = ca + \sum_1^m (cy_h - xd_h)a_h$. I.e. $c(y_n d - a - \sum_1^m y_h a_h) = x(d_n d - \sum_1^m d_h a_h)$ i.e. we have shown that if $K = \{r \in R \mid xr \in cR\}$ then K intersects $C(0)$ and $C(A)$ so by 2.4 $K \cap (C(A) \cap C(0)) \neq \emptyset$, i.e. $C(A) \cap C(0)$ is an Ore set.

(2) Same as (1) but simpler as we don't require the I in above to contain a regular element. □

CHAPTER FOUR - OVERINGS

§A. Localisations of e-νH-orders

In this section we wish to prove the new result that if R is an r-τ-Noetherian e-νH-order and if $R' = R_F = {}_GR$ for some Gabriel topology $F(G)$ of right (left) R-ideals of R then R' is an r-τ-Noetherian νH-order. We then look at the problem of when R' has enough ν-invertible ideals.

Lemma 4.1

If R is a τ-Noetherian (r-τ-Noetherian) additive regular order then if $I \in \tau_r$ (and $I \cap C(0) \neq \emptyset$) then there exists $K \subseteq I$, K a finitely generated right ideal (and $K \cap C(0) \neq \emptyset$) such that $K \in \tau_r$.

Proof

There exists K_R f.g. ($K \cap C(0) \neq \emptyset$) with $Cl_r(K) = Cl_r(I) = R$ (as $x(I:x)_R \subseteq I$ for any $x \in R$) thus $1H \subseteq K$ for some $H \in \tau_r$ thus $K \in \tau_r$. □

Proposition 4.2

If R is an r-τ-Noetherian (τ-Noetherian) additive regular order and if $R' = R_F = {}_GR \subseteq Q$ for some Gabriel topology $F(G)$ of right (left) R-ideals of R, then R' is an r-τ-Noetherian (τ-Noetherian) additive regular order.

Proof

This is analogous to 2.2(a) of [14]. We give a proof for completeness. We first claim that if $G \in F$ then $GR' \in \tau_r(R')$. For suppose $x \in R'$,

$y \in Q$ and $y(GR':x)_{R'} \subseteq R'$ then there exists $F \in \mathcal{F}$ with $xF \subseteq R$. Now if $f \in F$ then $f(G:xf)_R \subseteq (GR':x)_{R'}$ so $yf(G:xf)_R \subseteq R'$. By property (I) of $\mathcal{F}$ $\frac{Q}{R'}$ is $\mathcal{F}$-torsion free so $yf \in R'$ i.e. $yF \subseteq R'$ so $y \in R'$ i.e. $GR' \in \tau_r(R')$.

Now suppose $F \in \tau_r(R)$ and $F \cap C(0) \neq \emptyset$ then we claim $FR' \in \tau_r(R')$. For suppose $x \in R'$, $y \in Q$ and $y(FR':x)_{R'} \subseteq R'$ then there exists $G \in \mathcal{F}$ with $xG \in R$. Suppose $g \in G$ then $g(F:xg)_R \subseteq (FR':x)_{R'}$ so $yg(F:xg)_R \subseteq R'$. As $(F:xg)_R \in \tau_r$ and contains a regular element, by Lemma 4.1 there exist $I \in \tau_r$, I_R finitely generated, $I \cap C(0) \neq \emptyset$ and also $ygI \subseteq R'$. Therefore there exists $H \in \mathcal{G}$ with $HygI \subseteq R$ so $Hyg \subseteq R$ so $yg \in R'$ so $yG \subseteq R'$ so $y \in R'$ and we see that $FR' \in \tau_r(R')$. For the stronger case of when R is τ-Noetherian we see that the above holds for any $F \in \tau_r(R)$. Now suppose I' is a $\tau_r(R')$-closed right R'-ideal of R', then we will show that $I' \cap R$ is τ_r-closed and $I' = Cl_r((I' \cap R)R')$. This will clearly imply the result given in the proposition. Now if $x \in I'$ then there exists $F \in \mathcal{F}$ s.t. $xF \subseteq I' \cap R$, so $xFR' \subseteq (I' \cap R)R'$ so $x \in Cl_r((I' \cap R)R')$ i.e. $I' = Cl_r((I' \cap R)R')$. Also $I' \cap R$ is $\tau_r(R)$-closed for if $x \in Cl_r(I' \cap R)$ then there exists $F \in \tau_r$, $F \cap C(0) \neq \emptyset$ with $xF \subseteq I' \cap R$ so $xFR' \subseteq I'$. As $FR' \in \tau_r(R')$ $x \in I'$ so $x \in I' \cap R$. □

If I is a right ideal of an order R and $\mathcal{F}$ is a Gabriel topology we note that $I_{\mathcal{F}} = \{q \in Q | qF \subseteq I \text{ for some } F \in \mathcal{F}\}$.

<u>Lemma 4.3</u>

Let R be an additive regular order with the maximum condition on reflexive right (left)R-ideals of R. Suppose $R' = R_{\mathcal{F}} = {}_{\mathcal{G}}R \subseteq Q$ where $\mathcal{F}(\mathcal{G})$ is a Gabriel topology of right (left) R-ideals, then we have the following.

(1) If A is an integral R_F-ideal then $(R_F:A)_r = (R:(A \cap R))_{rF}$ and $O_r(A) \subseteq (O_r(A \cap R))_F$.

(2) If I is a right R-ideal then $_G(R:I)_\ell = (R:I_F)_\ell$ and $(I_F)_\nu = (I_\nu)_F$. If I is an integral reflexive right R_F-ideal of R_F then $I \cap R$ is a reflexive right R-ideal and $I = (I \cap R)_F$.

Proof

We prove (1), (2) is analogous see [14] 2.2(c). So suppose A is an integral R_F-ideal and $q \in (R_F:A)_r$, so $Aq \subseteq R_F$ so $(A \cap R)q \subseteq R_F$. Now there exists a f.g. left R-ideal B contained in $A \cap R$ with $_\nu B = {}_\nu(A \cap R)$, so we have $Bq \subseteq R_F$ and there exists $F \in F$ s.t. $BqF \subseteq R$. So $(A \cap R)qF \subseteq R$ and so $q \in (R:(A \cap R))_{rF}$. Conversely suppose $(A \cap R)qF \subseteq R$. Let $t \in A$ then there exists $G \in G$ with $Gt \subseteq A \cap R$, so $GtqF \subseteq R$ so $tqF \subseteq R_F$ and $tq \in R_F$ (using (I) property of F). Hence $Aq \subseteq R_F$ and so $q \in (R_F:A)_r$ i.e. we have $(R_F:A)_r = (R:(A \cap R))_{rF}$. Now suppose $q \in O_r(A)$ so $Aq \subseteq A \Rightarrow (A \cap R)q \subseteq A$. Now there exists a f.g. left R-ideal B contained in $A \cap R$ s.t. $_\nu B = {}_\nu(A \cap R)$. There exists $F \in F$ with $BqF \subseteq A \cap R \subseteq R$ so $(A \cap R)qF \subseteq R$ and hence $(A \cap R)qF \subseteq A \cap R$ i.e. $q \in (O_r(A \cap R))_F$. □

Corollary 4.4

If R is an additive regular r-τ-Noetherian (τ-Noetherian) e-νH-order then if $R' = R_F = {}_GR \subseteq Q$ for some Gabriel topology $F(G)$ of right (left) R-ideals of R then R' is an additive regular r-τ-Noetherian (τ-Noetherian) νH-order.

Proof

Only the last part requires proof. So suppose A is an integral R_F-ideal which is right R_F-reflexive. We have $(R_F:A)_r A = (R:(A\cap R))_{rF}(A\cap R)_F$. If $q(R:A)_r A \subseteq R_F$ then $q(R:(A\cap R))_r(A\cap R) \subseteq R_F$. Now there exists $B_R \subseteq (R:(A\cap R))_r(A\cap R)$, B_R a f.g. right R-ideal and $B_\nu = ((R:(A\cap R))_r(A\cap R))_\nu$. Then $qB \subseteq R_F$ so there exists $G \in G$ with $GqB \subseteq R$ so $q \in {}_G(R:(R:(A\cap R))_r(A\cap R))_\ell$. So if we let $C = (R:(R:(A\cap R))_r(A\cap R))_\ell$ then we have shown that $(R:(R_F:A)_r A)_\ell \subseteq {}_G C$.

Now if $y \in (R_F:{}_G C)_r = (R:C)_{rF} = ((R:(A\cap R))_r(A\cap R))_{\nu F} = (O_r(A\cap R))_F$ then ${}_G Cy \subseteq R_F$ so $y \in ((R_F:A)_r A)_\nu$ i.e. $(O_r(A\cap R))_F \subseteq ((R_F:A)_r A)_\nu \subseteq O_r(A) \subseteq (O_r(A\cap R))_F$ i.e. $O_r(A) = ((R_F:A)_r A)_\nu$ so A is left ν-projective as required. □

It seems an open problem whether or not in the situation of Cor. 4.4 R' has enough ν-invertible ideals, even when R is a hereditary Noetherian ring with enough invertible ideals. It is true when R' is the localisation of R with respect to an Ore set of regular elements say S. For given a ν-invertible R-ideal X of R, XR_S is a right reflexive R_S-ideal by 4.3(2). So if $s \in S$ then $s^{-n}XR$, $n \geq 1$ is an ascending chain of reflexive right R_S-ideals and we see $s^{-1}XR_S \subseteq XR_S$ and so XR_S is an ideal of R_S. Similarly R_SX is an ideal so $XR_S = R_SX$. Then $(R_S:XR_S)_\ell = R_S(R:X)_\ell$, $(R_S:XR_S)_r = (R:X)_rR_S$. As $R_S(R:X)_\ell s^{-n}$, $s \in S$, $n \geq 1$ is contained in the left R_S-ideal $R_S(R:X)_\ell R_S$ we see $R_S(R:X)_\ell \supseteq (R:X)_\ell R_S = (R:X)_rR_S \supseteq R_S(R:X)_\ell$. So by 2.9(3) $XR_S = R_SX$ is a ν-invertible R_S-ideal. It follows that R_S has enough ν-invertible ideals.

Another class of localisation for which it is true is bilateral localisations which we now look at.

We say a set of R-ideals H is ν-stable under multiplications if given $A,B \in H$ then there exists $C(D) \in H$ with $C \subseteq (AB)_\nu (D \subseteq {}_\nu(AB))$.

Proposition 4.5

Let R be an r-τ-Noetherian additive regular order and H a collection of R-ideals of R ν-stable under multiplication and satisfying the following condition.

(*) For each $A \in H$ there exists $C \in H$ $(D \in H)$ with $C(D)$ ν-invertible and $C \subseteq A_\nu$ $(D \subseteq {}_\nu A)$. Then let $R' = R(H) = \bigcup_{H \in H} (R:H)_\ell = \bigcup_{H \in H} (R:H)_r = \bigcup_{\substack{C \in H \\ C\ \nu\text{-invertible}}} C^{-1}$

then $R' = R_F = {}_G R$ for some Gabriel topology $F(G)$ of right (left) R-ideals of R.

If R is an e-νH-order then so is R'.

Proof

This result is essentially due to Chamarie note condition (*) is redundent for maximal orders.

Let F = class of right R-ideal I s.t. $(I:x)_{R\nu}$ contains a ν-invertible ideal in H for all $x \in R$. We prove this class coincides with F'_0 where F' is the right Gabriel topology cogenerated by $E(\frac{Q}{R'}) = E'$.

Let I be a right R-ideal of R with $\mathrm{Hom}_R(\frac{R}{I}, E') = 0$. Then $\mathrm{Hom}_R(\frac{R}{(I:x)_R}, \frac{Q}{R'}) = 0$ so $(R:(I:x)_R)_\ell \subseteq R'$. Let $K \subseteq (R:(I:x)_R)_\ell$ with K a f.g. left R-ideal and $(R:K)_r = ((I:x)_R)_\nu$, so $K(I:x)_{R\nu} \subseteq R$ and $K \subseteq R'$. So there exists X, X ν-invertible, $X \in H$ with $KX \subseteq R \Rightarrow x \subseteq (I:x)_{R\nu}$ i.e. $I \in F$. Conversely if $I \in F$ then to show $I \in F'_0$ one only need show

Hom $(\frac{R}{I}, \frac{Q}{R'}) = 0$. Suppose not and $qI \subseteq R'$ for some $q \in Q$, $q \notin R'$. Now there exists K, a f.g. right R-ideal with $K_\nu = I_\nu$ and $K \subseteq I$. There exists $X \in H$, X ν-invertible s.t. $XqK \subset R$ so $Xq(I)_\nu \subset R$; by hypothesis of F $I_\nu \supset Y$, Y a ν-invertible ideal and $Y \in H$ so $XqY \subseteq R$ i.e. $qYX \subseteq R$ i.e. $q \in R'$, this contradiction gives $F \subseteq F_0'$ and hence $F = F_0'$. Then clearly $R' = R_F$ similarly $R' = {}_GR$ for appropriate G. If R is an e-νH-order then by Cor. 4.4 we know R' is an r-τ-Noetherian νH-order. Also R' has enough ν-invertible ideals for suppose I is an integral R'-ideal which is right reflexive then $(R \cap I)$ is a reflexive ideal of R. Now $I \cap R \supseteq X$, X a ν-invertible ideal of R, then one easily checks $X_F = {}_GX$ and X_F is an ideal of R' also $(R:X)_{rF} = {}_G(R:X)_r$. Then Corollary 4.4 gives $(R_F:X_F)_r = (R_F:X_F)_\ell$ and by 2.7(3) X_F is ν-invertible. □

Let R be an additive regular order. We define the *Asano overing* $S(R) = S = \cup\{B^{-1} | B \text{ a } \nu\text{-invertible ideal of } R\}$. Note that S(R) is a ring containing R, for if A and B are ν-invertible ideals of R one easily sees $(AB)_\nu = {}_\nu(AB)$ is a ν-invertible ideal. In particular when R is r-τ-Noetherian then by 4.5 $S(R) = R_F = {}_GR$ for appropriate F, G.

Proposition 4.6

Let R be an r-τ-Noetherian additive regular order. If R has enough ν-invertible ideals then S(R) is a ν-simple Krull order.

Proof

We know from 4.2 and 4.5 that S(R) is r-τ-Noetherian. To show S(R) is ν-simple note that if A is an integral R-ideal of S(R) which is right reflexive then $A = (A \cap R)_F$ by 4.3(2) and $A \cap R$ contain a ν-invertible ideal X, so $A \supseteq XX^{-1}$ so $A \cap R \supseteq XX^{-1}$ so $A = (A \cap R)_{\nu F} \supseteq (XX^{-1})_{\nu F} = R_F = S(R)$. So $S(R)$ is ν-simple. This implies $S(R)$ is a maximal order for if A is any integral S-ideal and $qA \subseteq A \subseteq S(R)$ then $qA_\nu \subseteq S \Rightarrow q \in S$. □

Let R be an r-τ-Noetherian additive regular e-νH-order and P a maximal ν-invertible ideal of R, so $P = Q_1 \cap \ldots \cap Q_m$ where the Q_i are the maximal reflexive ideals containing P. We define:

$$R_{S(R-P)} = \left\{ \begin{array}{l} q \in Q \mid qX \subseteq R, \text{ for some } \nu\text{-invertible } X \not\subseteq Q_i \\ \text{for all } i. \end{array} \right\}$$

Then if H is the collection of R-ideals of R not contained in any Q_i we see H satisfies the condition (*) for else we would have a ν-ideal I of R maximal w.r.t. not containing any ν-invertible ideals except those contained in some Q_i. In particular I would be prime, I contains a ν-invertible ideal X and X is a product of maximal ν-invertible ideals so I contains a maximal ν-invertible ideal and hence $I \supseteq P$ this implies $I = Q_i$ for some i. Thus H satisfies condition (*).

<u>Corollary 4.7</u>

Let R be an additive regular r-τ-Noetherian e-νH-order. If P is a maximal ν-invertible ideal of R then

$R_{S(R-P)} = S(R) \cap R_{S(P)}$; $R_{S(R-P)}$ is an r-τ-Noetherian e-νH-order

$D(R_{S(R-P)}) \cong \mathbf{Z}$.

Also $R = \bigcap_{P} R_{S(R-P)}$, P = set of maximal ν-invertibles.

If P' is a non-empty subset of P then

$$R' = \bigcap_{P'} R_{S(R-P)} \text{ is an } r\text{-}\tau\text{-Noetherian } e\text{-}\nu H\text{-order.}$$

If $B = \bigcap_{P} R_{S(P)}$ then B is a regular r-τ-Noetherian e-νH-order.

Proof

The properties of $R_{S(R-P)}$ have been proved above. For the rest see [14]. □

CHAPTER FIVE - THE STRUCTURE OF $R_{S(A)}$.

From Chapter three we know that if R is an additive regular r-τ-Noetherian e-νH-order and A is a ν-invertible R-ideal of R then $S(A) = C(A) \cap C(0)$ is an Ore set. We would like to know about the structure of $R_{S(A)}$. One seems to need to assume R has a semi-local quotient ring in order to obtain any meaningful results.

Theorem 5.1

Let R be an order in a semi-local quotient ring. If R is an r-τ-Noetherian e-νH-order then if A is a ν-invertible R-ideal of R then $S(A) = C(A) \cap C(0)$ is an Ore set and $R_{S(A)}$ is a semi-local r-Noetherian r-hereditary regular ring. $\frac{R_{S(A)}}{K}$ is Artinian for any right $R_{S(P)}$-ideal K of $R_{S(A)}$ and $AR_{S(A)} = R_{S(A)}A$ is an invertible ideal of $R_{S(A)}$.

Proof

By 3.3 $C(A) = C(A_1) \cap \dots \cap C(A_h)$ where A_i are the maximal ν-invertible R-ideals of R containing A. Let $A_i = Q_{i1} \cap \dots \cap Q_{in_i}$ where the Q_{ij} are the maximal ν-ideals of R containing A_i.

We know there exists a semi-prime ideal H of R, $H = P_1 \cap \dots \cap P_n$ s.t. $C(0) = C(H) = C(P_1) \cap \dots \cap C(P_n)$. We order the P_i s.t. there exists $k > 0$ such that for $i < k$ $P_i \subseteq Q_{\ell,j}$ for some ℓ,j and for $i > k$ $P_i \not\subseteq Q_{\ell,j}$ for any ℓ,j. By $k = 0$ we mean no P_i is contained in any $Q_{\ell,j}$. We first show $R_{S(A)}$ is r-Noetherian. Let I be a right $R_{S(A)}$-ideal of $R_{S(A)}$ then $I = (I \cap R)R_{S(A)}$. Now $I \cap R$ is τ_r-closed for suppose $r \in Cl_r(I \cap R)$ then there exists $F \in \tau_r$ with $F \cap S(A) \neq \emptyset$ and $rF \subseteq I \cap R$ so $rs \in I \cap R$ for some $s \in S(A)$ so $r \in (I \cap R)s^{-1} \cap R \subseteq I \cap R$ so $I \cap R$ is τ_r-closed and it follows that $R_{S(A)}$ is r-Noetherian. As R has a semi-local quotient ring $P_iR_{S(A)} = R_{S(A)}P_i$, $i = 1,\dots,n$. Also if I is an R-ideal of R then if $c \in S(A)$ the chain $(c^{-n}IR_{S(A)})$ must stop so $IR_{S(A)}$ is an ideal of $R_{S(A)}$ and we see $IR_{S(A)} = R_{S(A)}I$. We next prove $R_{S(A)}$ is semi-local. By Lemma 2.5 we have

$$(\bigcap_{\ell,j} Q_{\ell,j} \cap P_{k+1} \cap \dots \cap P_n) \cap (C(P_1) \cap \dots \cap C(P_k)) \neq \emptyset.$$

Let $B = \bigcap_{\ell,j} Q_{\ell,j} \cap P_{k+1} \cap \dots \cap P_n$. If $c \in C(B)$ then $(cR + BR) \cap S(A) \neq \emptyset$ by 2.5 again. Also $S(A) \subseteq C(B)$ and $C(B) \cap C(0) = S(A)$. So we see $R_{S(A)}\ BR_{S(A)} \cap R = B$ and so $BR_{S(A)} = R_{S(A)}B$. Thus $\frac{R_{S(A)}}{BR_{S(A)}}$ is the quotient ring of $\frac{R}{B}$. Also $\frac{R}{B}$ is Goldie because we have the ring homomorphism $\frac{R}{B} \hookrightarrow \bigoplus_{\substack{i,j \\ \ell > k+1}} \frac{R}{Q_{ij}} \oplus \frac{R}{P_\ell}$. So $\frac{R_{S(A)}}{BR_{S(A)}}$ is Artinian. Now let M be a maximal right ideal of $R_{S(A)}$ we wish to show M contains $BR_{S(A)}$. If $M \not\supseteq BR_{S(A)}$ then $M + BR_{S(A)} = R_{S(A)}$ so $M \cap C(B) \neq \emptyset$. Now consider $i < k$ if $k \neq 0$. If $P_iR_{S(A)} \not\subseteq M$ then $M + P_iR_{S(A)} = R_{S(A)}$ so $M \cap C(P_i) \neq \emptyset$ as $C(0) \subseteq C(P_i)$. Also if $P_iR_{S(P)} \subseteq M$ then $\frac{M}{P_iR_{S(A)}}$ is a maximal right ideal of $\frac{R_{S(A)}}{P_iR_{S(A)}}$. If $M \cap C(P_i) \neq \emptyset$ then M is not essential in $\frac{R_{S(A)}}{P_iR_{S(A)}}$ so

$\frac{R_{S(A)}}{P_iR_{S(A)}}$ has a non-zero simple right ideal, so by 1.3 $\frac{R_{S(A)}}{P_iR_{S(A)}}$ is simple Artinian, a contradiction for $P_iR_{S(A)} \not\subseteq Q_{\ell,j} R_{S(A)}$ for some ℓ,j. So in fact we have shown $M \cap C(P_i) \neq \emptyset$. So we have shown that if $i < k$ then $M \cap C(P_i) \neq \emptyset$. So by 2.5 $M \cap S(A) \neq \emptyset$ thus $M = R_{S(A)}$ this contradicts the assumption $M \not\supseteq BR_{S(A)}$ so $M \supseteq BR_{S(A)}$ for all maximal right ideals of $R_{S(A)}$ so $J(R_{S(A)} = BR_{S(A)}$ and $R_{S(A)}$ is a semi-local ring. In particular $C(B)$ are units in $R_{S(A)}$ so $C(B) \subseteq S(A)$ i.e. $C(B) = S(A)$ i.e. $C(A) \cap C(0) = C(\bigcap_{\ell,j} Q_{\ell,j} \cap P_{k+1} \cap \dots \cap P_n)$. Now clearly $AR_{S(A)} = R_{S(A)}A$ is a ν-invertible ideal of $R_{S(A)}$. In particular $(AR_{S(A)})^{-1} AR_{S(A)}$ is not contained in any reflexive ideal of $R_{S(A)}$ and so in fact $AR_{S(A)}$ is invertible. By the Dual Basis lemma $AR_{S(A)}$ is projective. Now if M is a maximal right ideal of $R_{S(A)}$ containing a regular element then M contains $AR_{S(A)}$ for else $M \cap C(A) \neq \emptyset$ so $M \cap S(A) = \emptyset$. So M contains $AR_{S(A)}$. As $C(A) = \cap C(Q_{ij})$ we can assume A is semi-prime. Then $\frac{R_{S(A)}}{AR_{S(A)}}$ is semi-simple Artinian and so by Wedderburn's theorem there exists a right ideal N of $R_{S(A)}$ s.t. $M + N = R_{S(A)}$ and $M \cap N = AR_{S(A)}$. Define $\theta: M \oplus N \to R_{S(A)}$ by $\theta(m,n) = m-n$ then we have the exact sequence $0 \to AR_{S(A)} \to M \oplus N \to R_{S(A)} \to 0$ which splits as $R_{S(A)}$ is a projective $R_{S(A)}$-module, so $M \oplus N \cong AR_{S(A)} \oplus R_{S(A)}$ so M is a projective right $R_{S(A)}$-ideal.

Now suppose K is a right $R_{S(A)}$-ideal of $R_{S(A)}$ maximal with respect to not being projective then $\frac{R_{S(A)}}{K}$ is Artinian for if $I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \cdots \supseteq K$ is a descending chain of right ideals containing K then $(R_{S(A)}:I_0)_\ell \subseteq \cdots \subseteq (R_{S(A)}:K)_\ell$ must stop so $(I_n)_\nu = (I_{n+1})_\nu = \cdots$ but each I_j is projective by hypothesis and hence reflexive so $I_n = I_{n+1} = \cdots$ Therefore there exists a right ideal I containing K s.t. $\frac{I}{K}$ is a simple $R_{S(A)}$-module. So $\frac{I}{K} \cong \frac{R_{S(A)}}{M}$ for some maximal right ideal M. Now as $R_{S(A)}$ is semi-local there exists a maximal ideal C with $K \supseteq IC$. We want C to contain a regular element, if not then $C \cap C(A) \neq \emptyset$. Let $c \in I \cap C(0)$, so $c^{-1}K \cap R_{S(A)} \supseteq C$ so $(c^{-1}K \cap R_{S(A)}) \cap C(A) \neq \emptyset$.

Thus $(c^{-1}K \cap R_{S(A)}) \cap S(A) \neq \emptyset$ by additive regularity, so $c^{-1}K \cap R_{S(A)} = R_{S(A)}$ i.e. $c \in K$, but I is generated by it's regular elements so $K = I$, this contradiction means C and hence M contains a regular element. Therefore M is projective and by Schanuel's lemma $I \oplus M \cong K \oplus R_{S(A)}$ and so K is projective. Thus we have shown that $R_{S(A)}$ is r-hereditary. We now show $R_{S(A)}$ is regular. Let K be a right $R_{S(A)}$-ideal of R then $\frac{R_{S(A)}}{K}$ is Artinian so has a composition series. Above proof then gives that $AR_{S(A)}$ annihilates each simple factor and hence $K \supseteq A^n R_{S(A)}$ for some $n > 0$, i.e. $R_{S(A)}$ is regular. This concludes the proof of Theorem 5.1. □

<u>Proposition 5.2</u>

Let R be an order in a semi-local ring Q. If R is an r-τ-Noetherian e-νH-order then

(1) $R = \bigcap_{P \in \mathcal{P}} R_{S(P)}$ $\quad\mathcal{P}$ = class of maximal ν-invertible ideals of R.
$S(P) = C(P) \cap C(0)$.

(2) $S(R)$ is a ν-simple Krull order.

(3) If $c \in C(0)$ then $c \in C(P)$ for all but a finite number of $P \in \mathcal{P}$.

Proof

(3) is as in 1.8(b) of [14]. (2) is 4.6. We prove (1). Let $q \in$ R.H.S. then there exists a ν-invertible ideal X of R with $Xq \subseteq R$ choose X maximal w.r.t. this condition. If $X \neq R$ then $P \supseteq X$ for some maximal ν-invertible ideal P of R and $R \supseteq P^{-1}X \supsetneq X$ now $qc \in R$ for some $c \in S(P)$ so $PP^{-1}Xq \subseteq R \cap Pq \subseteq R \cap Pc^{-1} \subseteq P$ so $P^{-1}Xq \subseteq R$ so $(P^{-1} \circ X)q \subseteq R$. This contradicts the maximality of X so $q \in R$. □

Comment

One can obtain a converse of 5.2 and 5.1 as in 2.23 of [30] for the special case of R being prime Goldie. We do not offer a proof as it's length and importance do not justify it's inclusion.

As an application of the above we prove the following.

Recall an Artinian ring R is called *generalised uniserial* iff each indecomposable direct summand of the underlying right R-module R has a unique composition series; by 3.2 of [18] an Artinian ring R is generalised uniserial iff each left or right module is a direct sum of cyclic modules each of which has a unique composition series.

Proposition 5.3

If R is an r-τ-Noetherian e-νH-order in a semi-local quotient ring, then if A is a ν-invertible R-ideal of R then $\frac{R}{A}$ has a generalised uniserial Artinian quotient ring. If R is a Krull order then R/A has a principal ideal quotient ring.

Proof

The S(A) are units in $R_{S(A)}$ and we have $\frac{R}{A} \to \frac{R_{S(A)}}{AR_{S(A)}}$, $\frac{R_{S(A)}}{AR_{S(A)}}$ is Artinian so we see $Q(\frac{R}{A}) \cong \frac{R_{S(A)}}{AR_{S(A)}}$. Inspection of the proofs of 3.3 of [18] and Section 2 of [34] show that the analogous proofs carry through to the case of $R_{S(A)}$ with the properties proved in 5.1. $\square$

Proposition 5.4

Let R be an r-τ-Noetherian e-νH-order with semi-local quotient ring. If H is the semi-prime ideal s.t. C(0) = C(H), A a ν-invertible R-ideal of R with $Q_{\ell,j}$ and P_i given as in 5.1 then $\frac{R_{S(A)}}{HR_{S(A)}}$ is a semi-prime Noetherian hereditary ring or a semi-prime Artinian ring. In particular each $Q_{\ell j}$ contains at most one P_i.

Proof

If k = 0 in notation of 5.1 then $\frac{R_{S(A)}}{HR_{S(A)}}$ is semi-simple Artinian, else $P_i \subseteq Q_{\ell,j}$ for some ℓ,j,i. Then repeating an analogous proof of 5.1 gives the result noting that $AR_{S(A)}$ is still invertible in $\frac{R_{S(A)}}{HR_{S(A)}}$ (see 6.3). $\square$

CHAPTER SIX - THE STRUCTURE OF R_P AND THE RANKS OF THE MAXIMAL ν-INVERTIBLE IDEALS

§A. The structure of R_P

Lemma 6.1 [Chatters, Ginn]

Let R be a Noetherian semi-local ring s.t. J(R) is invertible and $J \cap C(0) \neq \emptyset$ then R is a semi-prime hereditary ring.

Proof

Theorem 1.5 of [16] shows R is hereditary. So by 1.9 R is the direct sum of prime rings and Artinian rings. If $A(R) \neq 0$ then $J(A(R)) \cap C_A(0) = \emptyset$ which contradicts the hypothesis so R is semi-prime hereditary. □

Lemma 6.2

Let R be an additive regular order in Q. If P is a localisable semi-prime ideal of R with $P \cap C(0) \neq \emptyset$ then $R_P \cong (R_{S(P)})_{R_{S(P)}PR_{S(P)}}$ where $S(P) = C(P) \cap C(0)$.

Proof

We have S(P) is an Ore set so $R_{S(P)}PR_{S(P)} \cap R = P$ so $R_{S(P)}PR_{S(P)} = PR_{S(P)} = R_{S(P)}P$ is a semi-prime ideal. It is easy to see

$C(PR_{S(P)}) = C(P)S(P)^{-1} = S(P)^{-1}C(P)$, and this implies $R \cap T(R_{S(P)}) = T(P)$ where $T(\_)$ denotes the appropriate torsion ideal. So

$$\frac{R}{T(P)} \hookrightarrow \frac{R_{S(P)}}{T(R_{S(P)}P)} \quad \text{and} \quad \frac{R_{S(P)}}{T(R_{S(P)}P)}$$

is the localisation of $\frac{R}{R(P)}$ w.r.t. $C(P) \cap C(0)$. The result follows. □

<u>Lemma 6.3</u>

Let R be an order in Q and let P be a localisable semi-prime ideal of Q with $P \cap C(0)) \neq \emptyset$ and let T be the corresponding torsion ideal of R, then if X is an invertible R-ideal of R containing T then $\frac{X}{T}$ is an invertible ideal of $\frac{R}{T}$ in the overring $\frac{Q}{TQ}$.

<u>Proof</u>

$T = \{r \in R \mid rc = 0 \text{ for some } c \in C(P)\}$. Clearly $C(0) \subseteq C(T)$ and so $QTQ \cap R = T = TQ \cap R$ so $TQ = QT$ and $\frac{R}{T} \to \frac{Q}{TQ}$. $\frac{Q}{TQ}$ is the localisation of $\frac{R}{T}$ w.r.t. $C(0)$. If X is an invertible R-ideal of R containing T then there exists $x_i \in X$, $y_i \in R$, $c \in C(0)$ s.t. $y_ic^{-1} \in X^{-1}$ and $1 = \sum_i x_i y_i c^{-1}$ so $c = \sum_i x_i y_i$.

Therefore $[TQ + c] = \sum_i [TQ + x_i]\,[TQ + y_i]$ where $[TQ + \_]$ denotes elements in the ring $\frac{Q}{TQ}$. So we have $1 = \sum_i [TQ + x_i]\,[TQ + y_i][TQ + c]^{-1}$.

Also let $x \in X$ then $xy_i \in Rc$ so $[TQ + x][TQ + y_i] \in [TQ + R][TQ + c]$ so $[TQ + y_i][TQ + c]^{-1} \in \{y \in \frac{Q}{TQ} \mid \frac{X}{T} y \in \frac{R}{T}\}$. So $\frac{X}{T}$ is right invertible, similarly on the left. □

Theorem 6.4

Let R be an additive regular τ-Noetherian e-νH-order then if $P = Q_1 \cap \dots \cap Q_m$ is a maximal ν-invertible ideal of R then P is localisable and R_P is a semi-local Noetherian hereditary semi-prime ring with $J(R_P) = PR_P$ an invertible ideal.

Proof

We first show P is localisable, by 3.10 we know C(P) is an Ore set so we only have to prove P is right (left) reversible. Let $I = \{a \in R \mid ac = 0 \text{ for some } c \in C(P)\}$. Then I is τ_r-closed for if $r \in Cl(I)$ then there exists $F \in \tau_r$ with $rF \subseteq I$. By 3.7 $F \cap C(P) \neq \emptyset$ so there exists $c \in C(P)$ s.t. $rc \in I$ so $r \in I$. Hence $\frac{R}{I}$ is right Goldie by 3.4. Suppose $c \in C(P)$ and $cr \in I$. Now there exists $n > 0$ with $(I:c^n)_R = (I:c^{n+1})_R$, also there exists $s \in R$, $d \in C(P)$ s.t. $c^n s = rd$ so $c^{n+1} = crd \in I$ so $c^n s \in I$ so $rd \in I$ so $r \in I$. Clearly if $c \in C(P)$ and $rc \in I$ then $r \in I$ so we have shown $C(P) \subseteq C(I)$. To prove right reversibility we note if we have $sa = 0$ for some $a \in R$, $s \in C(P)$ then $a \in I$ and so $ac = 0$ for some $c \in C(P)$.

We now note that R_P is Noetherian for let $T(P) = \{r \in R \; rc = 0$ for some $c \in C(P)\}$ and let I be a right ideal of R_P, $I \cap \bar{R} = \frac{I'}{T(P)}$

where $\bar{R} = \frac{R}{T(P)}$, $I = \overline{I'}\, R_P$. Then I' is τ_r-closed for if $rF \subseteq I'$, $F \in \tau_r$ then $rc \in I'$ for some $c \in C(P)$ so $\bar{r} \in \overline{I'}\, c^{-1} \cap \bar{R} = \overline{I'}$ so $r \in I'$ and so we see R_P is Noetherian. As in 5.1 we see that $PR_{S(P)} = R_{S(P)}$ is invertible so by 6.2 we can assume right away that the elements of $S(P)$ units and P is an invertible ideal. Now let $Y = \{$products of $C(P)$ and $C(0)\}$ then we claim Y is an Ore set in $\bar{R}$.

We prove this by induction on the lengths of elements of Y. If $n = 1$ then it is clear, as $C(P)$ and $C(0)$ are Ore sets, so suppose true for elements of Y of length less than or equal to n i.e. if $c = e_1 \ldots e_n$, $e_i \in C(P)$ or $C(0)$ and $r \in R$ then there exists $d \in Y$ and $s \in R$ s.t. $cs = rd$. Let $c = c_1 \ldots c_{n+1}$ where $c_i \in C(P)$ or $C(0)$ and let $t \in R$ then there exists $d \in Y$ and $r \in R$ s.t. $c_1 \ldots c_n\bar{r} = \bar{t}d$. Now $rx = c_{n+1}y$ for some $y \in R$, $x \in Y$ so $c_1 \ldots c_{n+1}\bar{y} = \bar{t}dx$ and $dx \in Y$ hence true for $n + 1$ and so Y is an Ore set of regular elements in $\bar{R}$. Let $K = R_Y$ then we have

$$\begin{tikzcd} & \frac{Q}{T(P)Q} \arrow[dr] & \\ \bar{R} \arrow[ur] \arrow[dr] & & K \\ & R_P \arrow[ur] & \end{tikzcd}$$

Now by 6.3 $\bar{P}$ is invertible in $\frac{Q}{T(P)Q}$ and hence in K. So $\bar{P}R_P = R_P\bar{P}$ is an invertible ideal of R_P in overring K. So by 6.1 R_P is a semi-prime hereditary ring. R_P is bounded by 1.9. □

Comment

In [28] Hudry has independently proved the above theorem for the case of τ-Noetherian maximal orders without the assumption of additive regular.

Close inspections of the proofs of Section 3 for this case seem to show one can get round the problems which additive regularity easily solve to show $C(P)$ is an Ore set for τ-Noetherian maximal orders. However for the r-τ-Noetherian case the additive regular condition seems indispensible. For general e-νH-orders one can probably get away from the additive regular condition in 6.4 by looking at $R[t]$ in $Q[t]$, as $R[t]$ is always an additive regular ring.

§B. The rank of maximal ν-invertible ideals.

Let P be a prime reflexive ideal of a prime Goldie maximal order and $0 \subsetneq Q \subsetneq P$, Q a prime ideal then $PP^{-1}Q \subseteq Q$ so $P^{-1}Q \subseteq Q$ and $P^{-1} = R$ a contradiction so P has rank one. For e-νH-orders this argument does not work so instead one looks at R_P or $R_{S(P)}$ and hence stronger chain conditions than Goldie have to be imposed. We consider the problem of ranks in the non-prime case.

Theorem 6.5

Let R be an r-τ-Noetherian additive regular e-νH-order. Let $P = Q_1 \cap \cdots \cap Q_m$ be a maximal ν-invertible ideal. Suppose we have for a fixed $i < m$ $\frac{R}{P'}$ Goldie for all prime ideals $P' \subseteq Q_i$ then

(1) If R is τ-Noetherian then Q_i has rank one.

(2) If $PR_{S(P)}$ is AR then Q_i has rank at most one.

Proof

(1) We know P is localisable and $C(P)$ is an Ore set. Let P' be a prime ideal contained in Q_i. Let $T'=\{r \in R \mid rc \in P' \text{ for some } c \in C(P)\}$ then T' is an ideal of R. Suppose $T' \not\subseteq P'$ then as $\frac{R}{P'}$ is Goldie there exists $e \in T'$ with $e \in C(P')$ (1.2). Then there exists $c \in C(P)$ s.t. $ec \in P'$ so $c \in P' \subseteq Q_i$ but $C(P) \subseteq C(Q_i)$. This contradiction shows $C(P) \subseteq C(P')$ and $T(P) \subseteq T' \subseteq P' \subseteq Q_i$. Now R_P is a semi-prime hereditary Noetherian ring so $Q_iR_P = R_PQ_i$ has rank one. Now suppose $\frac{P'}{T(P)}$ is a prime ideal of $\bar{R}$ contained in $\frac{Q_i}{T(P)}$ then $\frac{P'}{T(P)} R_P \cap \bar{R} = \frac{P'}{T(P)}$ for $C(P) \subseteq C(P')$ by above. We therefore have $P'R_P = R_PP'$ is a prime ideal of R_P. As we have shown any prime ideal contained in Q_i contains $T(P)$. Thus Q_i has rank one in R.

(2) As above if $P' \subseteq Q_i$ is a prime ideal we have $S(P) \subseteq C(P')$ and so $R_{S(P)}P'R_{S(P)} \cap R = P'$ and $P'R_{S(P)} = R_{S(P)}P'$ is a prime ideal of $R_{S(P)}$. So we only need show $Q_iR_{S(P)}$ has rank at most one.

Recall that an ideal I of a ring R is AR iff for every right ideal K there exists $n > 0$ s.t. $K \cap I^n \subseteq KI$ and similarly for the left ideals of R. By 3.3 of [2] an invertible ideal of a Noetherian ring is AR so the assumption that $PR_{S(P)}$ is AR given in the hypothesis is natural. We would now like to use the invertible ideal theorem as given in 3.4 of [2], but we here do not necessarily have $R_{S(P)}$ as a Noetherian ring. However $\frac{R_{S(P)}}{I}$ is Noetherian for any right $R_{S(P)}$-ideal I of $R_{S(P)}$. Also $PR_{S(P)}$ is AR. A close inspection of the proof of 3.4 of [2] shows in fact this is all we need and so we conclude that Q_i has rank at most one. $\square$

CHAPTER SEVEN - ARTINIAN QUOTIENT RINGS

In this section we prove a decomposition theorem and give applications for τ_o-Noetherian e-νH-orders which have Artinian quotient rings.

If R has an Artinian quotient ring then $C(0) = C(N)$ where N is the nilpotent radical of R. N is nilpotent and $N = P_1 \cap \ldots \cap P_n$ where the P_i are the minimal prime ideals of R.

Lemma 7.1

Let R have an Artinian quotient ring. Let P be a localisable semi-prime ideal containing a regular element with $\frac{R}{P}$ Goldie and such that $R_{S(P)}$ is Noetherian, then R_P has an Artinian quotient ring and R_P is a direct summand of $R_{S(P)}$.

Proof

Let $P = Q_1 \cap \ldots \cap Q_m$ where the Q_i are the minimal prime ideals over P and let $P_1, \ldots, P_n$ be the minimal prime ideals of R. As every prime ideal contains a minimal prime ideal of R, there exists $k > 1$ s.t. for $i < k$ P_i is contained in some Q_j and for $i \geq k$ no P_i is contained in any Q_j. Now suppose $i < k$ and $P_i \subseteq Q_j$ for some j then $C(P) \subseteq C(P_i)$, for let $T = \{r \in R | rc \in P_i$ for some $c \in C(P)\}$. If $T \not\subseteq P_i$ then as $\frac{R}{P_i}$ is Goldie there exists $e \in T \cap C(P_i)$ and so $ec \in P_i$ for some $c \in C(P)$ so $c \in P_i \subseteq Q_j$ a contradiction, so indeed $C(P) \subseteq C(P_i)$ and we have

$C(P) \cap C(0) = C(Q_1) \cap \dots \cap C(Q_m) \cap C(P_{k+1}) \cap \dots \cap C(P_n) =$ $C(Q_1 \cap \dots \cap Q_m \cap P_{k+1} \cap \dots \cap P_n)$. So $R_{S(P)}$ is semi-local with maximal ideals $Q_iR_{S(P)}$, $P_{k+1}R_{S(P)},\dots,P_nR_{S(P)}$. We can assume $k < n$ for else $C(P) \subseteq C(0)$ and $R_P = R_{S(P)}$ when the result is obvious. Now as $R_{S(P)}$ is Noetherian, $A = A(R_{S(P)})$ the Artinian radical of $R_{S(P)}$ is a direct summand of $R_{S(P)}$. Write $R_{S(P)} = A \oplus B$. Now for $i > k$ $P_iR_{S(P)}$ is a minimal prime ideal of $R_{S(P)}$ and $\dfrac{R_{S(P)}}{P_iR_{S(P)}}$ is simple Artinian. So by Lemma 4.10 of [2] $A \not\subseteq P_iR_i$ for $i > k$. In particular $A \neq 0$. Also $B \neq 0$ as not all prime ideals of $R_{S(P)}$ are minimal. We claim $B \cong R_P$. As $A \not\subseteq P_{k+1}R_{S(P)},\dots,P_nR_{S(P)}$, and the Q_i are not minimal we see $P_iR_{S(P)} = P_i' \oplus B$, $i = k+1,\dots,n$ for some prime ideals P_i' of A, and $Q_iR_{S(P)} = A \oplus Q_i'$ $i = 1,\dots,m$, for some prime ideals Q_i' of B. Clearly the P_i', $i = k+1,\dots,n$ are the prime ideals of A.

Then $A \subseteq PR_{S(P)}$ and $PR_{S(P)} = A \oplus (B \cap PR_{S(P)})$. So $\dfrac{R_{S(P)}}{PR_{S(P)}} \cong \dfrac{B}{B \cap PR_{S(P)}}$, so $\dfrac{B}{B \cap PR_{S(P)}}$ is a semi-prime Artinian ring. Also as $J(R_{S(P)}) = J(A) \oplus J(B)$ we see B is semi-local with maximal ideals Q_i' $i = 1,\dots,m$. Let $T = \{x \in R \mid xc = 0 \text{ some } c \in C(P)\}$. T is an ideal of R. Now $B \not\subseteq Q_iR_{S(P)}$ so $B \cap R \not\subseteq Q_i$ for all i. So $B \cap R$ contains an element of $C(P)$ as the $\dfrac{R}{Q_i}$ are Goldie by 1.15 applied to R/P. Now $(A \cap R)(B \cap R) = 0$ so $A \cap R \subseteq T$. Conversely, if $t \in T$ then $td = 0$ for some $d \in C(P)$ so $tdB = 0$ but if $d = e + f$ where $e \in A$, $f \in B$

then $f \in C_B$ $(B \cap PR_{S(P)})$ which are units of B. So dB = B and tB = 0 so $t \in A \cap R$. Hence $T = A \cap R$ so $\frac{R}{T}$ embeds in B and it is easily seen that B is obtained by localising $\frac{R}{T}$ at the semi-prime ideal P/T i.e. $B \cong R_P$ so $Q(R_P) \cong Q(B)$ is a direct summand of Q. □

Theorem 7.2

Let R be a τ_0-Noetherian e-νH-order in an Artinian quotient ring Q. Then R is the direct sum of prime τ-Noetherian e-νH-orders and a ring S which is a ν-simple strong Krull order.

Proof

We first show each $R_{S(P)}$ is Noetherian. Let I be a right ideal of $R_{S(P)}$ then we claim $I \cap R$ is τ_0-closed, for if $r \in Cl_{\tau_0}$ $(I \cap R)$ then $rF \subseteq I \cap R$ for some $F \in \tau_{r_0}$, $F \in \tau_r$ so $F \cap C(P) \neq \emptyset$ but $F \cap C(0) \neq \emptyset$ so $F \cap S(P) \neq \emptyset$, so $rc \in I \cap R$ for some $c \in S(P)$ so $r \in (I \cap R)c^{-1} \cap R \subseteq I \cap R$ so $I \cap R$ is τ_{r_0}-closed and hence $R_{S(P)}$ is Noetherian. Now suppose R is indecomposable and not semi-prime. Now $R = \bigcap_{P \in \mathcal{P}} R_{S(P)} \cap S(R)$ $\mathcal{P}$ = the set of maximal ν-invertible ideals of R.

And $R_{S(P)} \cong A(R_{S(P)}) \oplus R_P$ from proof of 7.1. If neither R_P nor $A(R_{S(P)})$ is zero then Q is not indecomposable. R_P is a semi-prime hereditary Noetherian ring and so is the direct sum of prime Noetherian rings, so if we write $Q = Q_1 \oplus \dots \oplus Q_\ell$ where the Q_i are indecomposable and $1 = e_1 + \dots + e_\ell$ where e_i is the identity of Q_i, we see Q_i is a direct summand of $A(R_{S(P)})$ or a direct summand of $Q(R_P)$ i.e. the e_i belong to $R_{S(P)}$. Now as $S(R)$ is a maximal order, by 1.17(1) we have the e_i belong to $S(R)$. Hence the e_i belong to R. We thus see that either $A(R_{S(P)}) = 0$ in which case $R_P \cong R_{S(P)}$ and so Q is semi-prime and hence so is R, or we must have the situation that R has no ν-invertible ideals i.e. $R = S(R)$. The case when R is semi-prime gives R as a direct sum of prime rings by the same argument. □

Comment

For the special case when R is a maximal order we can give a quicker proof by using 1.17. See [27].

We give applications of 7.2 at the end of the chapter.

Corollary 7.3

If R is a τ_0-Noetherian regular e-νH-order in an Artinian quotient ring then R is the direct sum of prime rings and Artinian rings.

Proof

We merely note that a regular ν-simple τ_0-Noetherian maximal order in an Artinian quotient ring is Artinian. □

Connected with the above is the proposition given below. For the definition of affiliated primes and their basic properties see Chapter 13 of [2].

The lemma below generalises 4.10 of [2].

Lemma 7.4

Let R be a Noetherian ring with a prime ideal P s.t. $\frac{R}{P}$ is Artinian and P does not contain a regular element then $A(R) \neq 0$; if P is minimal then $A(R) \not\subseteq P$.

Proof

By 13.7 of [2] there exists a right affiliated series $P_1, \ldots, P_m$, $0 = B_0 \subsetneqq B_1 \subseteq \cdots \subsetneqq B_m = R$ s.t. $B_k = A(R)$ for some $k > 0$ and $\frac{R}{P_i}$ is Artinian iff $i < k$.

Similarly there exists a left affiliated series $Q_1, \ldots, Q_\ell$ $0 = C_0 \not\subseteq C_1 \not\subseteq \cdots \not\subseteq C_\ell = R$. $C_t = A(R)$ etc. By 13.3 of [2] $\bigcap_{i,j} C(P_i) \cap C(Q_j) \subseteq C(0)$. It then follows that P must be an affiliated prime. Without loss of generality suppose $P = P_i$ for some $i < n$, then as $\frac{R}{P_i}$ is Artinian $k > i > 1$, so $A(R) \neq 0$. If P is minimal then $A(R) \subseteq P_i \Rightarrow r(A(R)) + A(R) \subseteq P$. But by 4.13 of [2] $(\ell(A) + A) \cap C(N) \neq \emptyset$. This contradiction gives $A(R) \not\subseteq P$ as required. □

Corollary 7.5

Let R be a τ_0-Noetherian e-νH-order with semi-local quotient ring. Let P be a maximal ν-invertible ideal. If R is not semi-prime then $A(R_{S(P)}) \neq 0$.

Proof

Let $H = P_1 \cap \cdots \cap P_n$, $P = Q_1 \cap \cdots \cap Q_m$ then some P_i is not contained in any Q_j or else $C(P) \subseteq C(0)$ and $R_{S(P)} = R_P$ is semi-prime by 6.4. For this i $\dfrac{R_{S(P)}}{P_i R_{S(P)}}$ is Artinian so we can apply 7.4. □

Corollary 7.6

Let R be a Noetherian order. Let P be a semi-prime ideal with $P \cap C(0) = \emptyset$ and also $C(P) \not\subseteq C(0)$ then S(P) is an Ore set and $A(R_{S(P)}) \neq 0$.

Proof

As above. □

Theorem 7.2 is probably the most important theorem of this paper as it seems to be a fundamental theorem. We give two applications below.

Proposition 7.7 (A. Chatters)

Let R be a Noetherian ring; if R satisfies the d.c.c. on prime ideals and every rank one prime ideal is principal i.e. of the form pR = Rp then R is the direct sum of prime Noetherian rings and a ring with no non-minimal prime ideals.

Proof

The proof is essentially due to A. Chatters (he avoids the use of S(P)). Let P = pR = Rp be a rank one prime ideal of R then $p \in C(N)$

for $px \in N \Rightarrow pRx \subseteq N$ so $x \in N$. Now $r(p^n) = r(p^{n+1})$ for some $n \geq 1$. If $px = 0$ for some $x \in R$ then $x \in N \subseteq pR$ so $x = px_1$ for some $x_1 \in R$, so $px = p^2x_1 = 0$ so $x_1 \in N$. Repeating we obtain a sequence x_m, $x_m \in R$, $m \geq 1$ with $x = p^m x_m$ and $p^{m+1}x_m = 0$ but $r(p^n) = r(p^{n+1})$ so $p^n x_n = x = 0$. This implies that the prime ideals that are maximal w.r.t. not containing a regular element are minimal, so by Corollary 2.15 of [36] R has an Artinian quotient ring. As $pR = Rp = P$ is invertible P is localisable by 1.3 of [16]. Thus $C(P) \cap C(0) = S(P)$ is an Ore set and $R_{S(P)} \cong A(R_{S(P)}) \oplus R_p$; R_p is prime Dedekind as $PR_p = R_pP$ is invertible. One easily proves that $R = \bigcap_P R_{S(P)} \cap S(R)$ and so R is a maximal order. Theorem 7.2 then gives the result. $\square$

We now use 7.2 and 7.3 to give a quick proof of the splitting of commutative or semi-perfect Noetherian FPF rings. Note that a ring is FPF if every f.g. faithful right (left) module is a generator of mod-R(R-mod). A ring is *semi-perfect* iff it is semi-local and idempotents can be lifted modulo the Jacobson radical. A module M_R is *balanced* if $R = \mathrm{End}_S M$ where $S = \mathrm{End}\, M_R$, see [5] and [17] for more details.

Proposition 7.8 (Endo, Faith)

Let R be a Noetherian FPF ring then if R is either semi-perfect or commutative then R is the direct sum of prime Dedekind rings and QF rings.

Proof

We first show R has a QF quotient ring. For the semi-perfect case this is true by Corollary 2.21 of [5] (the lack of semi-perfect in hypothesis is a misprint). For the commutative case either go through the first half of the proof of Endo's theorem in Theorem 11 of [4] p. 183 or quote the theorem that every commutative FPF ring has a self-injective quotient ring given in Theorem A, p. 72 of [3]. Also note R is a maximal order for a theorem of Morita (see 1.1D of [5]) states that a right module M generates mod-R iff M is projective over $B = \text{End } M_R$ and $R = \text{End}_B M$, i.e. for FPF rings every f.g. faithful right R-module is balanced. By 1.2(3) $\Rightarrow$ (1) of [17] we see R is a maximal order (for Noetherian rings one only needs M to be f.g. in (3) $\Rightarrow$ (2) of 1.2 of [17]). Thus by 7.2 R splits into a direct sum of prime Noetherian FPF's and a ring S where S is a Noetherian FPF with no reflexive ideals (see 3.4 of [5] for the summands being FPF). By 4.6 of [5] prime Noetherian FPF's are bounded Dedekind rings. If R is commutative the result is now immediate. If R is semi-perfect then S is semi-local but Morita's theorem above implies every integral S-ideal I is left projective over $\text{End } I_S \cong O_\ell(I) = S$ by 1.19. So every integral S-ideal is reflexive so S has no integral S-ideals and hence $J(S) = N(S)$ and so S is Artinian. $\square$

CHAPTER EIGHT - IDEALISERS

In this section we wish to prove the result that a prime Goldie τ-Noetherian e-νH-order with a finite number of ν-idempotent ideals is a finite intersection of Krull orders. We assume the reader is familiar with the definition of Idealisers. See [32] and [21] for details.

We define a ν-*semi-maximal* right ideal to be a reflexive right R-ideal of R which is the intersection of a finite number of maximal reflexive right R-ideals of R.

Lemma 8.1

Let R be a prime Goldie τ-Noetherian e-νH-order and K a regular reflexive right ideal of R. Let $I_0 = \{i \in I \mid KR_i \neq R_i\}$ and $A = \bigcap_{i \in \tau_0} A_i$ where the A_i are the maximal ν-invertible ideals of R for $i \in I$. Then K is ν-semi-maximal iff KR_A is semi-maximal.

Proof

This is 2.4 of [21]. □

Lemma 8.2

R as above, then if K is a regular ν-semi-maximal reflexive right R-ideal and $K' \supseteq K$ is reflexive then K' is a ν-semi-maximal right R ideal.

Proof

Let $I_0 = \{i \in I | KR_i \neq R_i\}$ $I_0' = \{i \in I | K'R_i \neq R_i\}$ $A = \bigcap_{i \in I_0} A_i$, $A' = \bigcap_{i \in I_0'} A_i$ then I_0', I_0 are finite and $I_0 \supseteq I_0'$ so $R_A \hookrightarrow R_{A'}$ and these rings are regular H.N.P. rings. $R_{A'}$ is the localisation of R_A at the ideal $A'R_A$. Now KR_A is semi-maximal by 8.1 so $K'R_A$ is semi-maximal so $K'R_{A'}$ is semi-maximal again by 8.1 applied to R_A and $K'R_A$. □

Lemma 8.3

R as above, I_0 a finite subset of I. K_i a semi-maximal right ideal of R_i, $i \in I_0$, then writing $K_i = R_i$ for $i \in I | I_0$ we have $K = \bigcap_I K_i \cap S(R)$ is a ν-semi-maximal right ideal and $I_R(K) = \cap\, I_{R_i}(K) \cap S(R)$. K is also regular.

Proof

We have $K_i \supseteq A_i$, $i \in I_0$, so $K \supseteq \bigcap_{i \in I_0} A_i$ so K is regular. Also $K_i \cap R$ is a reflexive right ideal by 4.3(2), so $K = \bigcap_{i \in I_0} (K_i \cap R)$ is reflexive. Also K is ν-semi-maximal for $\bigcap_{i \in I_0} A_i$ is ν-semi-maximal by 8.1 and $K \supseteq \bigcap_{i \in I_0} A_i$. Now $I_R(K) = \bigcap_{i \in I} I_{R_i}(KR_i) \cap S(R)$. If $j \in I_0$ then $K \supseteq (K_j \cap R)(\bigcap_{i \in I_0 | \{j\}} A_i)$ so $KR_j \supseteq (K_j \cap R)R_j = K_j$, so $KR_j = K_j$ so $I_{R_j}(KR_j) = I_{R_j}(K_j)$. □

Definition

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$ is a chain of right ideals of R then $I_R(A_1,\ldots,A_n) = \{r \in R \mid rA_k \subseteq A_k \text{ for each } k\} = \bigcap_{k=1}^{n} I_R(A_k)$. We call $I_R(A_1,\ldots,A_n)$ the multiple idealiser of R at $A_1,\ldots,A_n$. In [19] Ely shows that iterated idealisers at semi-maximal right ideals from a hereditary Noetherian prime ring can be obtained as a multiple idealiser from R at semi-maximal right ideals of R. Hence the following.

Proposition 8.4

If R is a τ-Noetherian prime Goldie e-νH-order then any iterated idealiser at regular ν-semi-maximal right ideals can be obtained as a multiple idealiser at regular ν-semi-maximal right ideals of R.

Proof

We have the following situation.

$T_0 = R = \bigcap_i R_i \cap S(R)$. Let K_0 be a ν-semi-maximal regular right ideal of R. We can assume $_\nu(RK_0) = R$. Then $T_1 = I_R(K_0) = \bigcap_i I_{R_i}(K_0R_i) \cap S(R) = \bigcap_i T_{1,i} \cap S(R)$. See proof of 2.9 of [21]. Also $T_{1,i}$ is a semi-local H.N.P. ring with $J(T_{1,i})$ being the unique maximal invertible ideal. T_1 is a prime Goldie τ-Noetherian e-νH-order. Repeating we obtain an iterated Idealiser: we have K_1 a ν-semi-maximal regular right ideal of T_1, $_\nu(T_1K_1) = T_1$.

$$T_2 = I_{T_1}(K_1) = \bigcap_i I_{T_{1,i}}(K_1T_{1,i}) \cap S(R)$$

$$T_n = I_{T_{n-1}}(K_{n-1}) = \bigcap_i I_{T_{n-1,i}}(K_{n-1}T_{n-1,i}) \cap S(R)$$

$$= \bigcap_i T_{n,i} \cap S(R)$$

Now $K_oR_i = R_i$ for all but a finite number of $i \in I$ and K_oR_i is a semi-maximal right ideal of R_i. Similarly for $K_1,\ldots,K_{n-1}$. Let I_o be the finite subset of I occurring for T_n i.e. $i \in I_o \leftrightarrow T_{n,i} \neq R_i$.

Now for $i \in I_o$, $T_{n,i}$ is an iterated idealiser of R_i at semi-maximal right ideals and hence is a multiple idealiser so there exists $A_{1,i} \subseteq A_{2,i} \subseteq \cdots \subseteq A_{n_i,i}$, $A_{j,i}$ a semi-maximal right ideal of R_i with $T_{n,i} = I_{R_i}(A_{1,i},\ldots,A_{n_i,i}) = \bigcap_{k=1}^{n_i} I_{R_i}(A_{k,i})$. Define $A_{m,i} = R_i$ for $m > n_i$ and $A_{k,i} = R_i$ for all k with $i \in I|I_o$. Let $A_t = \bigcap_{i\in I} A_{t,i} \cap S(R)$.

So $A_1 \subseteq A_2 \subseteq \cdots$. Then by Lemma 8.3 the A_t are regular ν-semi-maximal right ideals of R and $I_R(A_t) = \bigcap_{i\in I} I_{R_i}(A_{t,i}) \cap S(R)$. So $\bigcap_t I_R(A_t) = \bigcap_{t,i} I_{R_i}(A_{t,i}) \cap S(R) = \bigcap T_{n,i} \cap S(R) = T_n$. So T_n is also a multiple idealiser. □

The converse is also true by a similar argument but we do not need it.

Corollary 8.5

Let R be a prime Goldie τ-Noetherian e-νH-order then if R has a finite number of ν-idempotent ideals then R is the intersection of a finite number of Krull orders.

Proof

This should be compared with 4.9 of [18], but the proof there is not applicable. By 3.6 of [21] R is the iterated idealiser of regular ν-semi-maximal right ideals from a Krull order D say. Hence R is a multiple idealiser, so $R = I_D(A_1,\ldots,A_n)$ where the $A_1 \subseteq \cdots \subseteq A_n$ are regular ν-semi-maximal right ideals of D. So $R = \cap I_D(A_i) =$ $D \bigcap_{i=1}^{n} O_\ell(A_i)$. Now each $O_\ell(A_i)$ is a maximal order from 1.17(2). $O_\ell(A_i)$ is τ-Noetherian by Proposition 4.4 of [21]. The result follows. □

CHAPTER NINE - EXAMPLES

In [30] Marubayashi gives the following examples of prime Goldie τ-Noetherian e-νH-orders.

(1) Krull orders in the sense of [14].

(2) H.N.P. rings with enough invertible ideals.

(3) Tame orders over Krull domains.

(4) R[t], where R is a prime Goldie τ-Noetherian e-νH-order.

We wish to show further that the following are also e-νH-orders.

(5) Noetherian Homological homogeneous rings.

(6) K[G], where G is a torsion free polycyclic-by-finite group and K is a prime Noetherian maximal order of characteristic zero.

Note that (4) is the simplest way of producing an e-νH-order which is neither hereditary nor a maximal order.

Lemma 9.1

Suppose R is a prime Noetherian ring and such that $R = \cap R_i \cap S(R)$, where $S(R) = \cup\{B^{-1} | B$ a ν-invertible ideal of $R\}$ and the R_i are semi-local hereditary Noetherian prime rings with non-zero Jacobson radical. Then if R has enough ν-invertible ideals then R is an e-νH-order.

Proof

We will apply Lemma 1.1 of [21]. Firstly we note that S(R) is a ν-simple Krull order by 4.6. Also we can assume the $J(R_i)$ are the unique invertible ideals of R_i, for $S(R_i) = Q$ as R_i is bounded so $R_i = \bigcap_j (R_i)_{A_{j,i}}$ where the $A_{j,i}$ are the maximal invertible ideals of R_i and the $(R_i)_{A_{j,i}}$ are localisations of R w.r.t. Ore sets. We can thus apply Lemma 1.1(3) $\Rightarrow$ (1) if we show each $c \in C(0)$ is a unit of R_i for almost all $i \in I$. Now let $P_i = J(R_i) \cap R$ then P_i is a semi-prime reflexive ideal of R by 4.3. Now the units of R_i are precisely $C_{R_i}(J(R_i))$ so one easily sees the P_i are localisable and $R_i = R_{P_i}$.

Now we claim that if $P_i = Q_1 \cap \ldots \cap Q_n$, $P_j = Q_1' \cap \ldots \cap Q_m'$ $i,j \in I$ then $Q_\ell \neq Q_k'$ for all ℓ,k. It is easily seen that $R_{P_i \cap P_j} = R_{P_i} \cap R_{P_j}$ also we see $(P_i \cap P_j)R_{P_i \cap P_j}$ is ν-invertible and so $((P_i \cap P_j)R_{P_i \cap P_j}((P_i \cap P_j)R_{P_i \cap P_j})^{-1})_\nu = R_{P_i \cap P_j}$ and as all the $Q_\ell R_{P_i \cap P_j}$ and $Q_k' R_{P_i \cap P_j}$ are reflexive we see $(P_i \cap P_j)R_{P_i \cap P_j}$ is invertible and so $R_{P_i \cap P_j}$ is hereditary with $P_i R_{P_i \cap P_j}$, $P_j R_{P_i \cap P_j}$ the maximal invertible ideals of $R_{P_i \cap P_j}$ and so $Q_\ell \neq Q_k'$ for all ℓ,k. In particular $P_i R_j = R_j$ for all $i \neq j \in I$ so the P_i are ν-invertible as R has enough ν-invertible ideals (for then $qP_i \subseteq P_i \Rightarrow q \in S(R)$). Now suppose $c \in C(0)$, there exists a finite subset I_o of I s.t. $\sum_{i \in I} cP_i^{-1} \cap R = \sum_{i \in I_o} cP_i^{-1} \cap R$, so for $j \notin I_o$ $cP_j^{-1} \cap R \subseteq c \sum_{i \in I_o} (P_i^{-1} \cap c^{-1}R)$, so if $cx \subseteq P_j$ then $cxP_j^{-1} \subseteq R$ so $(\bigcap_{i \in I_o} P_i)c^{-1}\, cxP_j^{-1} \subseteq R$ so

$(\bigcap_{i\in I_o} P_i)x\, P_j^{-1}\, P_j \subseteq P_j$ and above then implies $x\, P_j^{-1} \subseteq P_j$ so $xP_j^{-1} \subseteq R$, so $x \in P_j$ similarly for $xc \in P_j$ i.e. $c \in C(P_j)$ i.e. c is a unit in R_{P_j} for $j \notin I_o$. Thus R satisfies all the hypotheses of Lemma 1.1 of [21] (3) $\Rightarrow$ (1). □

Definition

A Noetherian ring R is called Homologically homogeneous iff it has finite global dimension, is integral over it's centre Z and if V and W are simple right R-modules such $r_Z(V) = r_Z(W)$ then pr $\dim_R V$ = pr $\dim_R W$. See [11] for more details of Hom hom rings. Hom hom rings generalise Noetherian local rings of finite global dimension which are integral over their centres. Those rings are maximal orders by [24]. Hom hom rings however do not need to be maximal orders but in fact are e-νH-orders.

Proposition 9.2

Noetherian Hom hom rings are e-νH-orders.

Proof

By 5.3 of [11] R is the direct sum of prime Hom hom rings so we can assume R is prime. As R is integral over it's centre Z one sees R is fully bounded and every ideal contains a central regular element. Hence S(R) = Q and R has enough ν-invertible ideals. If P is a rank one prime ideal in Z then there exists only a finite number of rank one prime ideals of R lying over P. By Lemma 5.1 of [11] we have $R = \bigcap_{P \in I} R_P$ I = set of rank one prime ideals of Z and each R_P is hereditary.

By 3.7 of [13] $J(R_P) \cap Z_P = J(Z_P) = PZ_P \neq 0$ and so by 1.9 R_P is semi-local and $J(R_P) \neq 0$. Hence by 9.1 R is an e-νH-order. □

Corollary 9.3

Let G be a finitely generated group with an abelian normal subgroup of finite index. If G is torsion free then $\mathbb{Z}G$ is an e-νH-order or if k is a field with characteristic $p > 0$ and G has no element of order p then kG is an e-νH-order.

Proof

Proposition 7.5 of [11]. □

In [29] it is proved that if G is a torsion free polycyclic-by-finite group then $\mathbb{Z}G$ is a maximal order. This relies on the theorem that if R is the integral group ring of a torsion-free polycylic-by-finite group then the f.g. projective R-modules are stably free. This theorem is used to show that various localisations of R which are semi-local hereditary rings are in fact Dedekind. As we do not need this for e-νH-orders we naturally obtain the following.

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By 3.7 of [13] $J(R_P) \cap Z_P = J(Z_P) = PZ_P \neq 0$ and so by 1.9 R_P is semi-local and $J(R_P) \neq 0$. Hence by 9.1 R is an e-νH-order. □

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Proof

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Proposition 9.4

Let J be a prime Noetherian maximal order of characteristic zero and G a torsion free polycyclic-by-finite group. Then $J[G] = R$ is a Noetherian e-νH-order.

Proof

We will apply 9.1. We will use the notation of [29]. Let H be the characteristic subgroup of finite index in G s.t. H is poly-(infinite cyclic) given in [29]. Let $S = JH$ then S is a maximal order with quotient ring $Q(S) = Q$. If T is a transversal for H in G then $R = \bigoplus_{t\in T} S\,t$. with quotient ring $Q' = \bigoplus_{t\in T} Qt$.

Now suppose I is an ideal of R which is right reflexive then $I \cap S \neq 0$ (see [29], but best seen by observing that R is a normalising extension of S and so we can use 4.16 of [35]). Also $I \cap S$ is a G-invariant ideal of S and is clearly reflexive. $A = \bigoplus_{t\in T} (I \cap S)t = \bigoplus_{t\in T} t(I \cap S)$ is a ν-invertible ideal of R contained in I and hence R has enough ν-invertible ideals. It is clear that $S(R) = \bigoplus_{t\in T} S(S)t = \bigoplus_{t\in T} tS(S)$. In [29] it is shown that if P is a reflexive prime ideal of S and if $\tilde{P} = \bigcap_{t\in T} P^t$ then $C_S(\tilde{P})$ is an Ore set of regular elements in R and $\bar{R} = R_{\tilde{P}} = R_{C_S(\tilde{P})} = \bigoplus_{t\in T} S_{\tilde{P}}t$. If $I = R_{C_S(\tilde{P})}\tilde{P} = \bar{R}\tilde{P}$ then it is shown that I is a semi-prime invertible ideal of $\bar{R}$ and in Theorem 10 it is shown that $I = J(\bar{R})$ and hence $\bar{R}$ is a semi-local hereditary ring. It is also clear that $R = \bigcap_{P \in \mathcal{P}} R_{\tilde{P}} \cap S(R)$ $\mathcal{P}$ = reflexive prime ideals of S.

Hence we can use 9.1 to obtain R as an e-νH-order. □

A simple way to produce non-prime maximal orders is the following.

<u>Proposition 9.5</u>

Let R be a Noetherian maximal order and M a localisable semi-prime ideal of R containing a regular element such that $\frac{R}{M}$ is a maximal order with no reflexive ideals. Let $S = C_R(M) \cap C_R(0)$, $\bar{R} = \frac{R}{M}$ then

$$T_1 = \begin{bmatrix} R & \bar{R} \\ 0 & R \end{bmatrix} \text{ is a maximal order in } \begin{bmatrix} R_S & (\bar{R})_S \\ 0 & R_S \end{bmatrix}$$

$$T_2 = \begin{bmatrix} R & \bar{R} \\ 0 & R \end{bmatrix} = \left\{ \begin{bmatrix} x & \bar{y} \\ o & x \end{bmatrix}, x \in R, y \in R \right\} \text{ is a maximal order in}$$

$$\begin{bmatrix} R_S & (\bar{R})_S \\ 0 & R_S \end{bmatrix}.$$

If $C(M) \subseteq C(0)$ then

$$T_3 = \begin{bmatrix} R & \bar{R} \\ 0 & \bar{R} \end{bmatrix} \text{ is a maximal order in } \begin{bmatrix} R_S & (\bar{R})_S \\ 0 & (\bar{R})_S \end{bmatrix}.$$

If R has no reflexive ideals then

$T_4 = \begin{bmatrix} R & R \\ 0 & R \end{bmatrix}$ is a maximal order in $\begin{bmatrix} Q & Q \\ 0 & Q \end{bmatrix}$.

Proof

Consider T_1 if $\begin{bmatrix} c_1 & \bar{a} \\ 0 & c_2 \end{bmatrix}$ is regular in T, then c_1 is right regular, c_2 is left regular. Also if $bc_1 = 0$ for some $b \in R$ then $\begin{bmatrix} eb & o \\ o & o \end{bmatrix} \begin{bmatrix} c_1 & \bar{a} \\ o & c_2 \end{bmatrix} = 0$ where $e \in M \cap C(0)$ so $b = 0$, similarly c_2 is right regular. Also c_1 is right regular mod M and hence by 1.13 of [2] $c_1 \in C(M)$, similarly for c_2. The converse is clear i.e. $C_{T_1}(0) = \left\{\begin{bmatrix} c_1 & \bar{a} \\ o & c_2 \end{bmatrix}, c_1, c_2 \in S, a \in R\right\}$.

Also for such c_1, c_2 $\begin{bmatrix} c_1 & \bar{a} \\ o & c_2 \end{bmatrix} \begin{bmatrix} c_1^{-1} & -c_1^{-1}\bar{a}c_2^{-1} \\ o & c_2^{-1} \end{bmatrix} = \begin{bmatrix} c_1^{-1} & -c_1^{-1}\bar{a}c_2^{-1} \\ o & c_2^{-1} \end{bmatrix} \begin{bmatrix} c_1 & \bar{a} \\ o & c_2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and we see $Q(T_1) = \begin{bmatrix} R_S & (\bar{R})_S \\ 0 & R_S \end{bmatrix}$.

Now suppose F is an order in $Q(T_1)$ containing T_1 then F has the form $\begin{bmatrix} K & L \\ 0 & W \end{bmatrix}$ where $K^2 = K$, $W^2 = W$, $KL + LW \subseteq L$, $K, L, W \supseteq R$. If $\begin{bmatrix} K & L \\ 0 & W \end{bmatrix} \begin{bmatrix} c_1 & \bar{a} \\ 0 & c_2 \end{bmatrix} \subseteq T_1$ for some $c_1 \in S$, $c_2 \in S$, $\bar{a} \in \bar{R}$ then we have $Kc_1 \subseteq R$, $Wc_2 \subseteq R$, $K\bar{a} + Lc_2 \subseteq \bar{R}$.

As R is a maximal order $K = R = W$ and so $Lc_2 \subseteq \bar{R}$, but L is also a $\bar{R}$-$\bar{R}$-module and $\bar{R}$ has no reflexive $\bar{R}$-ideals hence $L = \bar{R}$. So $F = T_1$. Similarly for an order $F \supseteq T_1$ left equivalent to T_1 and hence by 1.16 T_1 is a maximal order.

Similar arguments work for T_2, T_3, T_4. □

Example 1

Put $R = \mathbb{Z}$, $M = 2\mathbb{Z}$, $F = \frac{\mathbb{Z}}{2\mathbb{Z}}$ then by above $T = \begin{bmatrix} \mathbb{Z} & F \\ 0 & F \end{bmatrix}$ is a maximal order. It has prime spectrum:

$$\begin{tikzcd}[column sep=small]
\cdots & \begin{bmatrix} 5\mathbb{Z} & F \\ 0 & F \end{bmatrix} & \begin{bmatrix} 3\mathbb{Z} & F \\ 0 & F \end{bmatrix} & \begin{bmatrix} 2\mathbb{Z} & F \\ 0 & F \end{bmatrix} & \begin{bmatrix} \mathbb{Z} & F \\ 0 & 0 \end{bmatrix} \\
 & & & \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix} \arrow[ull, dash] \arrow[ul, dash] \arrow[u, dash] &
\end{tikzcd}$$

The maximal zero divisor ideals are $\begin{bmatrix} 2\mathbb{Z} & F \\ 0 & F \end{bmatrix}$ and $\begin{bmatrix} \mathbb{Z} & F \\ 0 & 0 \end{bmatrix}$.

Example 2

Take $R = \mathbb{Z}$, $M = 2\mathbb{Z}$, $F = \mathbb{Z}/2\mathbb{Z}$ then

$$T = \begin{bmatrix} \mathbb{Z} & F \\ 0 & \mathbb{Z} \end{bmatrix} \text{ is a maximal order in } \begin{bmatrix} \mathbb{Z}_{(2)} & F \\ 0 & \mathbb{Z}_{(2)} \end{bmatrix} \text{ and}$$

the prime spectrum of T is

$$\begin{tikzcd}[column sep=small]
\cdots & \begin{bmatrix} 3\mathbb{Z} & F \\ 0 & \mathbb{Z} \end{bmatrix} & \begin{bmatrix} 2\mathbb{Z} & F \\ 0 & \mathbb{Z} \end{bmatrix} & \begin{bmatrix} \mathbb{Z} & F \\ 0 & 2\mathbb{Z} \end{bmatrix} & \begin{bmatrix} \mathbb{Z} & F \\ 0 & 3\mathbb{Z} \end{bmatrix} & \cdots \\
 & & \begin{bmatrix} 0 & F \\ 0 & \mathbb{Z} \end{bmatrix} \arrow[ul, dash] \arrow[u, dash] & \begin{bmatrix} \mathbb{Z} & F \\ 0 & 0 \end{bmatrix} \arrow[u, dash] \arrow[ur, dash] & &
\end{tikzcd}$$

Then $\begin{bmatrix} 2\mathbb{Z} & F \\ 0 & \mathbb{Z} \end{bmatrix}$, $\begin{bmatrix} \mathbb{Z} & F \\ 0 & 0 \end{bmatrix}$ are the maximal zero divisor primes.

<u>Example 3</u>

A commutative example.

Take $R = \mathbb{Z}$, $M = 2\mathbb{Z}$, $F = \frac{\mathbb{Z}}{2\mathbb{Z}}$.

Then $T = \begin{bmatrix} \mathbb{Z} & F \\ 0 & \mathbb{Z} \end{bmatrix}$ is a commutative maximal order with prime spectrum.

$$\begin{tikzcd}[column sep=small] \begin{bmatrix} 2\mathbb{Z} & F \\ 0 & 2\mathbb{Z} \end{bmatrix} \arrow[d, dash] & \begin{bmatrix} 3\mathbb{Z} & F \\ 0 & 3\mathbb{Z} \end{bmatrix} \arrow[dl, dash] & \begin{bmatrix} 5\mathbb{Z} & F \\ 0 & 5\mathbb{Z} \end{bmatrix} \arrow[dll, dash] & \cdots \\ \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} & & & \end{tikzcd}$$

$\begin{bmatrix} 2\mathbb{Z} & F \\ 0 & 2\mathbb{Z} \end{bmatrix}$ is the unique maximal zero-divisor ideal.

<u>Example 4</u>

As mentioned in the introduction and Chapter 3 Small's example $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ is a maximal order with an Artinian quotient ring. It is an r-τ-Noetherian ring but it is not τ-Noetherian. The prime ideals $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & p\mathbb{Z} \end{bmatrix}$ are reflexive but they are not localisable, however $C(P) \cap C(0)$ is an Ore set.

Example 5

An example of a Noetherian maximal order f.g. as a module over it's centre but it's centre is not a maximal order.

Let $S = \mathbb{Z} + \mathbb{Z}_{(3)}Y$ where Y is a commutating indeterminate with $Y^2 = 0$.

Let $P = 3\mathbb{Z} + \mathbb{Z}_{(3)}Y$

so $\frac{S}{P} \cong \mathbb{Z}_3$.

Let $R = \begin{bmatrix} S & \frac{S}{P} \\ 0 & \frac{S}{P} \end{bmatrix}$ then $Q(R) = \begin{bmatrix} S_P & \frac{S}{P} \\ 0 & \frac{S}{P} \end{bmatrix}$

Then it is easily checked that R is a maximal order. Now $Z(R) = \begin{bmatrix} r & o \\ o & r \end{bmatrix}$, $r \in S$. So $Z(R) \cong S$. But S is not a maximal order for $3(\mathbb{Z} + \frac{\mathbb{Z}_{(3)}}{3} Y) \subseteq S$ where $\frac{\mathbb{Z}_{(3)}}{3} = \{x \in Q \mid 3x \subseteq \mathbb{Z}_{(3)}\}$.

Z(R) is however a maximal order under the hypotheses of Example 5 if in addition we assume R has an Artinian quotient ring.

Example 6

An example of a Noetherian maximal order in an Artinian quotient ring with an equivalent order which does not split.

Let $R = \{(n,m),\ n-m \in 2\mathbb{Z},\ n,m \in \mathbb{Z}\}$ then $Q(R) = Q \oplus Q$ and R is equivalent to $\mathbb{Z} \oplus \mathbb{Z}$ which is a maximal order but R does not split.

Example 7

An example of an indecomposable Noetherian maximal order in an Artinian ring which is neither prime nor Artinian and which has rank one prime ideals none of which are reflexive or localisable.

Let B be the Noetherian integral domain constructed in [12] Example 7.2 chosen so that the global dimension of B is greater than one. B is a local ring whose Jacobson radical J is the unique rank one prime ideal of B. The ring B is constructed as a localisation at a particular prime ideal of a group ring kG of a poly-(infinite cyclic) group G. To show kG is a maximal order one uses [7] Corollary 2.6, p. 95 and notes that kG is produced by a succession of twisted polynomial ring constructions as given in [23] Example 4. Thus B is a maximal order and $J^* = R$. Let $R = \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}$ then by 9.5 R is a maximal order and R has the desired properties.

Open problems

(1) If R is a Noetherian e-νH-order with $C(0) = C(H)$ then is $\frac{R}{H}$ an e-νH-order? In particular the case when R is a Noetherian maximal order with an Artinian quotient ring.

(2) If R is a Noetherian eνH-order and $Q(R) = Q_1 \oplus \dots \oplus Q_n$ does R split as well.

(3) As mentioned in Chapter 4 if R is an additive regular r-τ-Noetherian e-νH-order and $R' = R_F = {}_G R$ for some Gabriel topology $F(G)$ of right (left) R-ideals then we know R' is an r-τ-Noetherian νH-order but does R' have enough ν-invertible ideals?

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